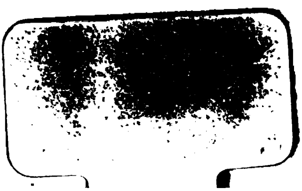

GEOMETRY

ELEMENTS

OF

GEOMETRY

BY

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LONDON
C. KEGAN PAUL & CO., 1 PATERNOSTER SQUARE
1881

ADVERTISEMENT.

THIS TRANSLATION was made from the Edition of Clairaut's 'Élémens de Géométrie,' published at Paris in 1830. It has had the advantage of being carefully corrected by Mr. JOHN IMRAY, M.A., to whom the Translator desires to express his indebtedness.

PREFACE.

ALTHOUGH Geometry be in itself abstract, it must be admitted that the difficulties felt by those who commence its study most frequently arise from the manner in which it is taught in ordinary elementary works. They generally begin with a great number of definitions, postulates, axioms, and preliminary principles, which seem to promise nothing but dryness to the reader. As the propositions which follow do not fix the mind upon the most interesting objects, and as they are besides difficult to understand, it commonly happens that beginners become wearied and discouraged before they get any distinct idea of what it is desired to teach them.

It is true that to avoid the dryness naturally belonging to the study of Geometry, some authors have stated after every important proposition the practical use to which it can be applied, but while they thus prove the utility of Geometry they do not facilitate its study; for as every proposition comes before its application the mind does not return to concrete objects till after it has undergone the fatigue of conceiving abstract ideas.

Reflections that I have made upon the origin of Geometry have led me to hope that these drawbacks might be avoided by uniting the two advantages of interesting and enlightening beginners.

I have thought that this science, like all others, must have become formed by degrees; that probably some want led to the necessity for taking the first steps, and that these first steps could not have been beyond the range of beginners, since it was by beginners that they were taken.

With this idea before me I have resolved to go back to that which may have given birth to Geometry, and I have endeavoured to develop its principles by a method such as may naturally be supposed to be that of its first inventors, taking care, however, to avoid the false attempts which they necessarily had to make.

The measurement of land has appeared to me that which most probably gave rise to the first propositions of Geometry, and this was really the origin of the science, since Geometry means *measurement of the earth.* Some authors pretend that the Egyptians, continually seeing the boundaries of their properties destroyed by the overflow of the Nile, laid the foundations of Geometry, so that they might have means of exactly ascertaining the situation, extent, and form of their domains.

But without regard to these authors there can be little doubt that from the earliest times men have sought means of measuring and dividing their lands. Wishing afterwards to improve these methods, particular researches gradually led them to general researches, and in their final determination to know the exact relations of all kinds of magnitudes they formed a science of a scope far more extended than they had at first intended, for which they retained the name that had been originally given to it.

In order to follow in this work a track similar to that of the inventors, I intend at the outset to give beginners means of discovering the principles on which simple measurement of lands and of accessible and inaccessible distances, &c., may be made to depend.

Thence I will proceed to other researches of a nature so analogous to the first, that the curiosity natural to all men must lead them so far; and by afterwards gratifying that curiosity by some useful application, I will proceed to discuss all the most interesting subjects of Elementary Geometry.

It cannot I think be denied that this method is at least likely to encourage those who might be repelled by the dryness of geometrical truths when stripped of their application, but I hope that it will have besides higher use, in that it may accustom the mind to seek and to discover; for I will carefully abstain from giving any propositions under the form of theorems (that is to say, of those propositions by which truths are demonstrated) unless I show at the same time how this discovery of the truth has been arrived at.

If the first mathematical authors presented their discoveries in the form of theorems it was doubtless for the purpose of giving a more marvellous character to their productions, or of avoiding the trouble of restating the train of ideas which had guided them in their researches. However that may be, it has appeared to me more judicious to employ my readers constantly in solving problems, that is to say, in seeking the means of per-forming some operation or of discovering some unknown truth by determining the relation which exists between given magnitudes and magnitudes unknown which it is desired to discover.

In following this course beginners will perceive at each step that they are made to take the reasons which guide the discoverer, and thus they can acquire more easily the spirit of discovery.

I may perhaps be reproached in some parts of these Elements for referring too much to the evidence of the eyes and for not keeping sufficiently close to rigorous exactitude of demonstration. I would ask those who might so reproach me to observe that I pass lightly only over those propositions of which the truth is discovered with a moderate degree of attention.

I do this chiefly at the beginning, where propositions of this kind are more frequently met with, for I have remarked that those who have a turn for Geometry are pleased by a little exercise of their minds; and that, on the other hand, they are disheartened when they are overwhelmed with demonstrations which, so to speak, are useless.

It is not to be wondered at that Euclid takes the trouble to demonstrate that two circles which cut each other have not the same centre;—that a triangle enclosed within another has the sum of its sides less than that of the sides of the triangle in which it is enclosed. That Geometer had to convince obstinate sophists who gloried in refusing to admit the most evident truths; it was necessary then that Geometry should, like Logic, have the aid of formal reasoning in order to close the mouths of sceptics. But things have changed. All reasoning that proves what common sense decides beforehand is now mere waste, and serves only to obscure truth and disgust students. I may also be reproached for having omitted various propositions which are found in ordinary elementary works, and for contenting myself with giving only the fundamental principles of propositions.

To this I answer that this treatise contains all that is necessary for my purpose; that the propositions which I omit are of no use in themselves, and give no aid in understanding those which it is important to know; that what I say in respect of proportions should suffice to render intelligible the elementary theorems which refer to them. That is a matter which I shall treat more fully in the Elements of Algebra which I intend to publish hereafter.

To conclude, as I have chosen the measurement of land in order to interest beginners, ought I not to fear that these Elements may be confounded with the ordinary treatises on land surveying? This thought can only come to those who do not consider that the measurement of land is not the real object of this book, but that it serves as the means of discovering the chief truths of Geometry. I might, indeed, have gone back to these truths by tracing the history of Physics, of Astronomy, or of any other branch of mathematics that I might have thought proper to choose; but in that case the multitude of foreign ideas which would necessarily have then engaged attention would have stifled the purely geometrical ideas on which only I had to fix the mind of the reader.

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ELEMENTS OF GEOMETRY.

FIRST PART.

OF THE MEANS THAT IT WAS MOST NATURAL TO EMPLOY FOR THE MEASUREMENT OF LAND.

It seems that the first things to be measured must have been lengths and distances.

1.

To measure any length, the expedient furnished by a sort of natural geometry, is to compare the length of a known measure to a length that it is desired to know.

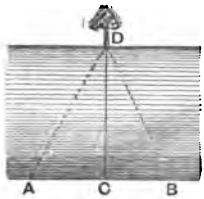
2.

It is evident that in order to measure the distance between two points, a straight line must be drawn from the one to the other, and that the known measure must be applied to this line, because all other lines, making necessarily a circuit more or less great, are longer than the straight line which makes no deviation.

3.

Besides measuring the distance from one point to another, it is often necessary to measure the distance from a point to a line. A man, for example, placed at D, upon the bank of a river, desires to know how far he is from the other bank A B. It is clear that, in order to measure this distance, he must select the shortest of all the possible lines, D A, D B, etc., that can be drawn from the point D to the straight line A B. It is readily seen that this shortest line is D C, which does not seem to incline either towards A or towards B. It is therefore to this line, which is named *perpendicular*. that the known measure is to be applied, in order to ascertain the distance D C from the point D to the straight line A B. But in order to apply this measure to the line D C, that line must first be drawn. It was therefore necessary to have a method of drawing perpendiculars.

Fig. 1.



4.

There are many other cases in which perpendiculars are required. It is evident, for instance, that the regularity of figures such as A B C D, F G H I, called *rectangles*, and composed of four sides perpendicular to each other, commended them as the proper forms for houses, gardens, chambers, walls, etc.

Fig. 2.

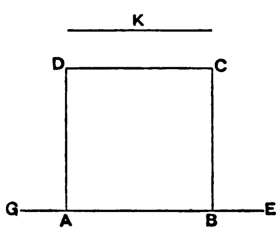
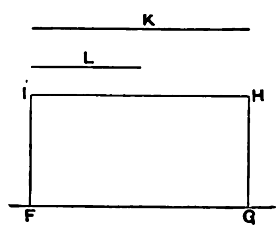


Fig. 3.



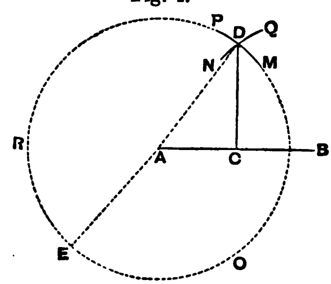
The first of these figures A B C D having the four sides equal, is commonly called a *square*. The other F G H I, which has only its opposite sides equal, retains the name of rectangle.

5.

In the different operations which require perpendiculars to be drawn, they have either to be let fall upon a line from a point outside of it, or to be erected from a point in the line itself.

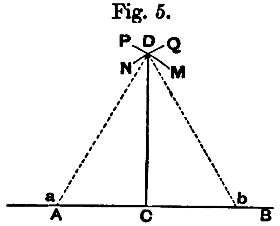
If from the point C in the line A B it is desired to erect a line C D, perpendicular to A B, this line must incline neither towards A nor towards B. Supposing then in the first place that C is equally distant from A and B, and that the line C D does not incline to either side, it is obvious that every point in this line must be equally distant from A and from B. It is only necessary then to find some point D such that its distance from A is equal to its distance from B, and to draw through C and through this point a straight line C D, which is the perpendicular required.

Fig. 4.



The point D might be found by trial; but trial does not satisfy the mind: it demands a precise method, which is as follows. Take an ordinary measure, a cord, for instance, or a compass opened to a certain width, according as you have to work upon land or upon paper.

Fig. 5.



Fix at the point A either the one end of the cord, or the one point of the compass, and, making the other end or point sweep round it, trace the arc P D M; then, without altering the measure, perform the same operation from the point B as a centre, so as to describe the arc Q D N, which, cutting the first arc at the point D, gives the point required.

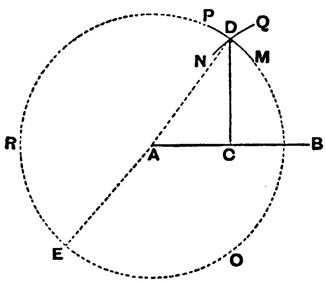
For since the point D belongs equally to both the arcs P D M, Q D N, described by a radius of the same length, its distance from the point A must equal its distance from the point B. Accordingly, as C D does not incline towards A nor towards B, this line is perpendicular to A B.

If the point C be not equally distant from A and from B, it is only necessary to take two other points *a* and *b* equally distant from C, and use them instead of A and B as centres for describing the arcs P D M, Q D N.

6.

If some of the arcs drawn, such as P D M, be continued through O, E and R, till it returns to the point P where it started, the entire trace is called the *circumference of a circle*, or simply a *circle*.

Fig. 6.



If only a part P D M of the circumference be drawn, that part is called an *arc of a circle*. The fixed point A is its *centre*, or the centre of the circle.

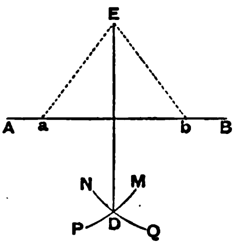
And the distance A D is its *radius*.

Every line like D A E, which passes through the centre A and which terminates at the circumference, is called a *diameter*. It is obvious that this line is double the radius, and hence the radius is sometimes called the *semi-diameter*.

7.

The method of erecting a perpendicular to a line A B leads to that for letting fall a perpendicular from any point E, outside of the line; for, by placing at E either the end of a cord or the point of a compass, and with an equal distance E *b* marking two points *a* and *b* upon the line A B, it is only necessary to find, by the previous method, another point D, equally distant from *a* and *b*, and through this point and E to draw the straight line D E, which, having each of its extremities equally distant from *a* and from *b*, and therefore not inclining more towards one of these points than towards the other, is perpendicular to A B.

Fig. 7.



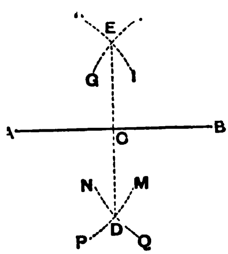
8.

From the preceding operation is derived the solution of a new problem.

Let it be desired to bisect a straight line A B, that is, to divide it into two equal parts.

From the points A and B as centres, and with any opening of the compass or radius, describe arcs R E I, G E F, from the same centres A and B, and with the same, or with any other radius, describe also arcs P D M, Q D N. Then the line E D, which joins the points of intersection E and D, cuts A B in two equal parts at the point C.

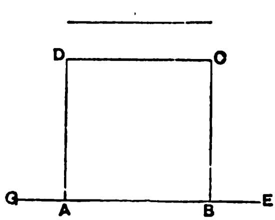
Fig. 8.



9.

The method of drawing perpendiculars being found, nothing was more easy than to use it for constructing the figures called *rectangles*, and *squares*, referred to in Par. 4. To make a square A B C D, of which the sides are to be equal to the given line K, take on the straight line G E a distance A B equal to K, then erect (Par. 5) at the points A and B the perpendiculars A D, B C, each equal to K, and join D C.

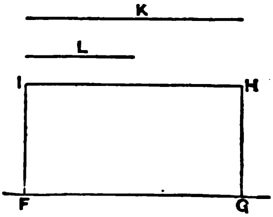
Fig. 9.



10.

To draw a rectangle F G H I of length K and width L, make F G equal to K, erect the perpendiculars F I and G H each equal to L, then join H I.

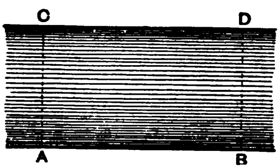
Fig. 10.



11.

In the construction of works such as ramparts, canals, streets, etc., it is necessary to draw parallel lines, that is to say, lines such that the distance between them perpendicularly measured is everywhere the same. To draw such parallels, nothing seems to be more natural than to adopt the method for drawing rectangles. Let A B, for instance, be one side of a canal, or rampart, etc., which is to be of the width C A, or, to express the problem in a more fitting and geometrical way, let it be supposed that through C is to be drawn C D parallel to A B. Taking any point B in the line A B, and operating in the same manner as for drawing on the base A B, a rectangle A B D C, having A C for its height,—then the lines C D, A B, if they were prolonged to infinity, would be always parallel, or, what comes to the same thing, they could never meet.

Fig. 11.



12.

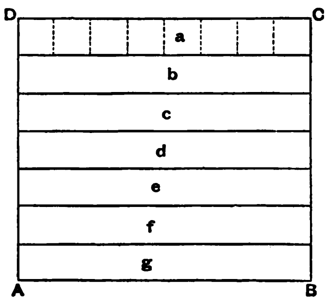
As the regularity of rectangular figures causes them to be frequently employed, as already stated, there are many cases where it is necessary to know their extent. It may be required, for instance, to determine how much carpet is wanted for a room, or how many acres are contained in a rectangular enclosure.

To determine such questions, the simplest and most natural method seems to be to make use of a commonly adopted measure, which, applied repeatedly over the surface to be measured, should cover it entirely, a method corresponding with that employed for determining the lengths of lines.

It is obvious, however, that the measure of surfaces must itself be a surface, such as a square yard, or a square foot, etc. Thus, to measure a rectangle is to determine the number of square yards or square feet that its surface contains.

To make the meaning clear by an example, let the given rectangle A B C D have seven feet of height, upon a base of eight feet; this rectangle may be considered to be divided into 7 bands *a*, *b*, *c*, *d*, *e*, *f*, *g*, each containing 8 square feet. The area of the rectangle must therefore be 7 times 8 square feet, or 56 square feet.

Fig. 12.



On recalling the first Elements of arithmetical calculation, it may be remembered that, to multiply two numbers is to take the one as many times as unity is contained in the other, and thus a perfect analogy subsists between ordinary multiplication and the operation of measuring a rectangle. It is seen that by multiplying the number of yards or of feet, etc., in its height, by the number of yards or of feet, etc., in the base, the number of square yards, or square feet, etc., contained in the surface, is determined.

13.

The figures which have to be measured are not always regular like rectangles, but their measurement is nevertheless often required.

Sometimes it may be necessary to determine the extent of a work constructed upon irregular ground; sometimes it may be necessary to know the acreage of a field, having an irregular boundary, etc.

It was therefore necessary that to the method of determining the extent of rectangles should be added that of measuring figures which are not rectangular.

It is at once seen that, practically, the difficulty applies only to the measurement of rectilineal figures, such as A B C D E, that is to say, of figures bounded by straight lines, for if curves should occur in the outline, as in the figure A B C D E F G, it is obvious that these curved lines, when divided into as many parts as may be necessary to avoid sensible error, may always be considered an assemblage of straight lines. This being under

Fig. 13.

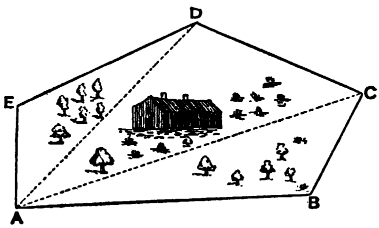
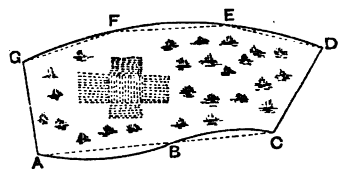


Fig. 14.



stood, it is seen that, notwithstanding the infinite variety of rectilineal figures, they can always be measured in the same way by dividing them into three-sided figures, commonly called *triangles*, which may be done in a simple and convenient way, by drawing from any point A, of the figure A B C D E, straight lines A C, A D, etc., to the other points C, D, etc.

14.

It will then only be necessary to measure the triangles so formed. In order to discover what is unknown, the surest way is to seek among what is known for something that may have relation to that which it is desired to discover. Now it has already been shown that every rectangle A B C D is equal to the product of its base A B by its height C B. Further it is easy to perceive that this figure, when cut across by the line A C, named its *diagonal*, is divided in two equal triangles; and hence it is inferred that each of these triangles is half the product of its base A B, or D C, by its height C B, or D A.

It seldom happens, however, that the triangles to be measured have two of their sides perpendicular to each other like the triangles A B C, A D C, which are called right-angled triangles; but there is nothing to prevent us from reducing all triangles to triangles of the same kind.

For if from the vertex A of any triangle A B C a perpendicular A D be let fall upon the base B C, then the triangle A B C is divided into two right-angled triangles, A D B, A D C. To resume, then, it is evident that as the two triangles A D B, A D C are the halves of the

Fig. 15.

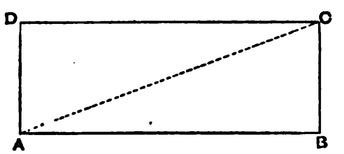
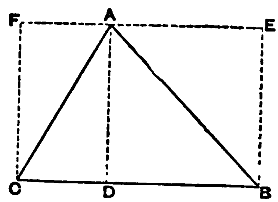


Fig. 16.



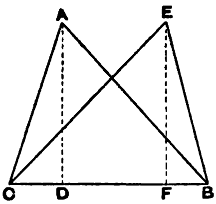
rectangles A E B D, A D C F, the whole triangle A B C must be likewise half of the whole rectangle E B C F, having B C for its base and A D for its height. But since the surface of the rectangle E B C F is the product of the height E B or A D by the base B C, the triangle A B C is measured by half the product of the base B C by the perpendicular A D or height of the triangle.

Thus are furnished means of measuring all lands bounded by straight lines, since there are none that cannot be reduced into triangles, and from the vertices of these triangles perpendiculars can be let fall upon their bases.

15.

As in the method thus given for measuring the area or surface of triangles only their base and their height are considered,

Fig. 17.



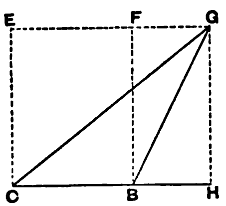
without regard to the length of their sides, it follows as a proposition, or theorem, that all triangles such as E C B, A C B, having a common base C B and equal heights E F, A D, have the same area.

16.

To render it more easy to understand the principle of measuring triangles, that side has been chosen for base upon which a perpendicular can be let fall from the opposite vertex—a choice that can always be made in measuring lands.

But as, in comparing triangles which have the same base, the perpendiculars let fall from their vertices may fall outside of the triangle, as in figure 18, it may be necessary to ascertain if triangles such as B C G follow the same law as the others, that is to say, if they also are halves of the rectangles E C B F which have the perpendicular G H for their height.

Fig. 18.

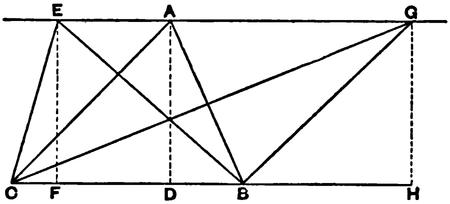


This is readily ascertained by observing that the triangle C G H, which is the sum of the two triangles C G B, G B H, is half of the rectangle E C H G, which is the sum of the two rectangles E C B F, F B H G, that is to say, the two triangles C G B, G B H, taken together, are half of the rectangle E C H G. Now the triangle G B H is the half of the rectangle F B H G, therefore the triangle C G B is half of the other rectangle E C B F, which has B C for its base, and G H for its height.

17.

The proposition demonstrated in the three preceding paragraphs may be generally expressed in these terms: the triangles E B C, A B C, G B C are equal, when they have a common base B C, and when they are between the same parallels E A G, C B H, that is to say, when their vertices E, A, G, are in one straight line E A G

Fig. 19.

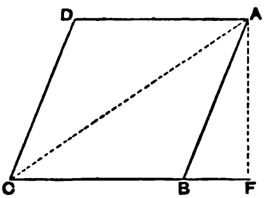


parallel to the straight line C B, for then (Par. 11) their heights, measured by the perpendiculars E F, A D, G H, are equal.

18.

Amongst the different rectilineal figures that can be measured by the preceding method, there are some that approach to the regularity of rectangles, such as the spaces A B C D bounded by four sides, of which each is parallel to the side opposite to it. These figures are called parallelograms, and are more easy to measure than any other rectilineal figures, except rectangles. For if the parallelogram A B C D be divided into two triangles, A B C, A C D, these two triangles are visibly equal; and, as each of these triangles is half the product of the height A F by the base B C, the parallelogram must be measured by the whole product of the base B C by the height A F.

Fig. 20.



19.

It follows that all parallelograms A B C D, E B C F, having a common base and lying between the same parallels, are equal. This may be readily seen, even independently of what precedes, on observing that the parallelogram A B C D would become the parallelogram E B C F if the triangle D C F were added to it, and the triangle A B E cut off from it; and if the two triangles D C F, A B E are equal, it is obvious that the parallelogram A B C D cannot change in area or extent in becoming E B C F. To make

Fig. 21.

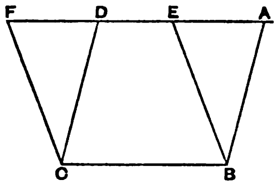
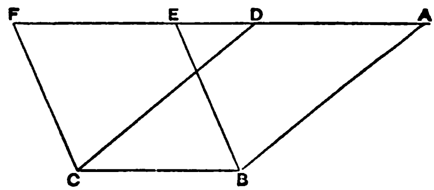


Fig. 22.



sure of the equality of these two triangles, it is sufficient to observe that A B and C D, being parallel, as well as B E and C F, the triangle D C F can be nothing else than the triangle A B E moved along its base, till the point A reaches D and E reaches F.

20.

There are also other rectilineal figures easy of measurement

Fig. 23.

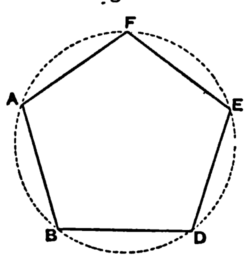
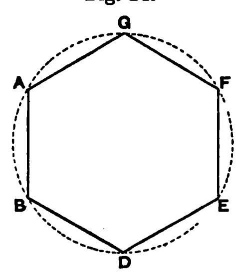


Fig. 24.



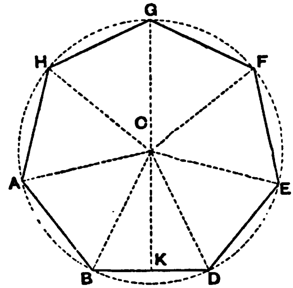
which are called regular polygons—figures bounded by equal sides, having all the same inclination to one another. Such are the figures A B D E F, A B D E F G, A B D E F G H.

As it is usual to give the symmetrical form of these figures to basins, fountains, to public places, etc., it would be well, before learning how to measure them, to see how they can be drawn.

21.

Having described the circumference of a circle and divided it into as many equal parts as there are sides to the proposed polygon: draw the lines A B, B D, D E, etc., through the points A, B, D, E, etc., which divide the circumference forming the polygon required, which is called a pentagon, a hexagon, a septagon, an octagon, a nonagon, or a decagon, etc., according as it has five, six, seven, eight, nine or ten sides, etc.

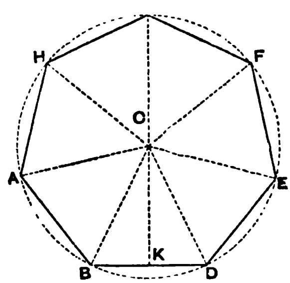
Fig. 25.



22.

In order to measure a regular polygon, the method already given (Par. 13) for all rectilineal figures may be employed; but it is readily seen that the shortest way is to divide the polygon into equal triangles, all having the centre C for their common vertex. Taking one of the triangles, C B D for example, and drawing on the base B D the perpendicular C K which in that case is named the *apotheme* of the polygon: as the area of the triangle is the product of the base B D by half of C K, this product taken as many times as the polygon has sides gives the area of the entire figure.

Fig. 26.

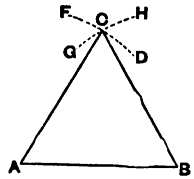


23.

If the circumference of the circle were divided into only three equal parts, a triangle would be formed commonly called an *equilateral triangle*.

If the circumference were divided into four equal parts, a square would be formed; but these two figures, the simplest of all polygons, can easily be drawn, without having recourse to the division of a circle, as has been already seen (Par. 9) for the square.

Fig. 27.



With respect to the equilateral triangle, it is readily seen that, in order to describe it upon a given base A B, the points A and B are to be taken as centres, and with an opening of the compass equal to A B, arcs D C F and G C H are to be described, and lines A C, B C, drawn from A and B to C, the intersection of the two arcs D C F, G C H, and the vertex of the triangle.

24

To the method of describing geometrically the equilateral triangle and the square, the simplest of all polygons, might be added that of drawing a pentagon, as many authors have done in their Elementary treatises; but as beginners, to whom alone this treatise is addressed, would have great difficulty, in following the road which Algebra has pointed out, for drawing this figure, it may be well to postpone the description of the pentagon to another period in a treatise following this, in which that description will be given as well as that of all the other polygons having a greater number of sides which, without the help of Algebra, cannot be described geometrically.

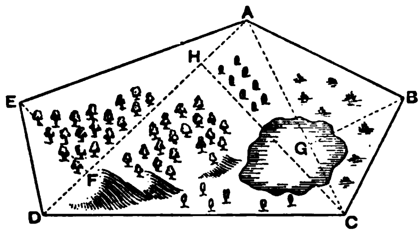
From the polygons having more than five sides that can only be described by means of Algebra must be excepted those of 6, 12, 24, 48, etc., sides, and those of 8, 16, 32, 64, etc., sides, which can easily be described by methods furnished by elementary geometry, as will be seen at the close of this first part.

25.

Returning to the measurement of lands, there are many cases where the forms are such that their measurement cannot be effected by the methods already described.

Let A B C D E be the figure of a field or an enclosure, etc., which it is desired to measure. Following what has preceded, let it be divided into triangles such as ABC, ACD, ADE, and let these triangles be measured after having drawn the perpendiculars EF, CH, BG. But if in the space ABCDE there should be some obstacle, an elevation, for example, a wood, a pond, etc., which prevents the necessary lines from being drawn, what then must be done? What method must be adopted in order to get over the inequality of the soil? The first expedient that presents itself to the mind is to choose some flat ground where one can easily operate, and on it to describe triangles equal and similar to the triangles ABC, ACD, etc. Let us see how such new triangles are to be formed.

Fig. 28.



26.

Supposing in the first place that the obstacle occurs within the triangle ABC, of which the sides are known, and that an equal and similar triangle is to be drawn upon the chosen ground. First we should draw a line DE equal to the side AB; then, taking a cord of the length BC and fixing one of its ends in E, we should describe the arc IFG, which will have the cord for radius, and by means of another cord taken equal to AC, and attached by one end at D, we should draw the arc KFH cutting the former arc in the point F. Then drawing the lines DF and FE, there is a triangle DEF equal and similar to the proposed triangle ABC. This is evident, for the sides DF and EF, which meet at the point F, being respectively equal to the sides AC and BC meeting at the point C, and the base DE having been taken equal to AB, it is not possible that the position of the lines

D F and E F upon D E should be different from the position of the lines A C and B C upon A B. It is true that the lines D *f*, E *f*,

Fig. 29.

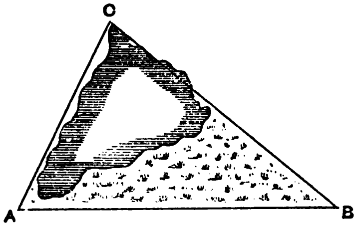
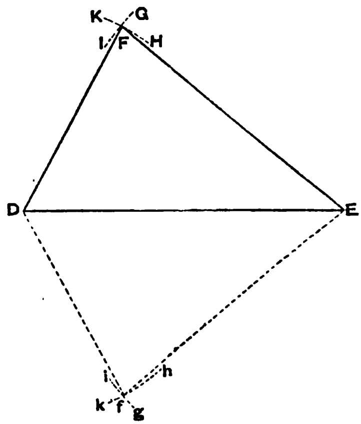


Fig. 30.



might be drawn below D E, but the triangle so produced would be still the same: it would be simply reversed.

27.

If only two of the sides of the triangle A B C, A B, B C, for instance, could be measured, it is clear that with this alone we could not determine a second triangle equal and similar to A B C; for though we might take D E equal to B C, and D F equal to B A, we could not know what position to give to the one relatively to the other. A simple expedient presents itself for overcoming the difficulty: it is to

Fig. 31.

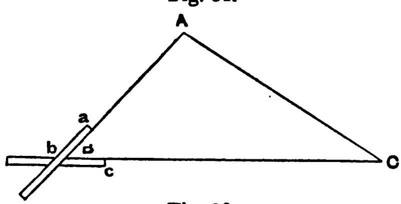
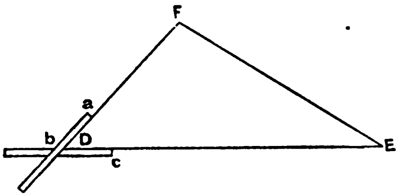


Fig. 32.



make D F incline in the same manner upon D E as A B inclines upon B C or, to express it geometrically, to make the angle F D E equal to the angle A B C.

28.

To perform this operation, we take an instrument such as *a b c*, composed of two bars which can be turned round *b*, and we put the bars on the sides A B and B C so that they contain between them the same angle as the sides A B and B C. Placing then the bar *b c* upon the base D E, with the centre *b* on the point D and keeping the angular opening of the instrument always the same, the bar *a b* will give the position of the line D F so as to make, with the line D E, the angle F D E equal to the angle A B C. Now the line D F having been taken of the same length as B A, it is only necessary to draw through F and E the straight line F E in order to have the triangle F E D completely equal and similar to the triangle A B C. This simple problem takes for granted this obvious principle, that a triangle is determined by the length of two of its sides, and by their angular opening; or what amounts to the same, that one triangle is equal to another when two of their sides are respectively equal and when the angle contained by those sides is the same.

29.

The angle F D E might be made equal to the angle A B C in the following manner.

Fig. 33.

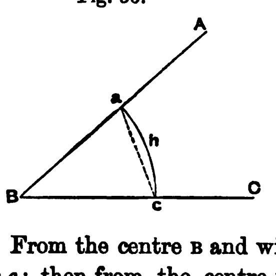
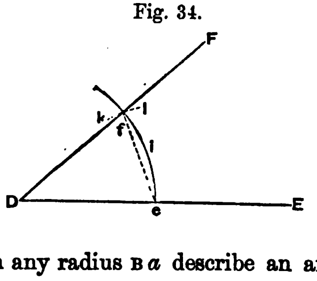


Fig. 34.



From the centre B and with any radius B *a* describe an arc *a h c*; then from the centre D with the same radius draw the arc *e i f*; then it is only necessary to find a point *f* situated on the arc *e i f* in the same manner as *a* is situated upon the arc *c h a*. This point *f* is easily found by means of the straight line *a c* which, according to the ordinary definition, is called *the chord of the arc a h c.*

For if, from the centre *e* and with a radius equal to *a c* the arc *l f k* be drawn, then the intersection of the two arcs *e i f*, *l f k*, gives the required point *f*.

Drawing then through D and *f* the line D*f* F, the angle F D E is equal to the angle A B C since (Par. 26) the triangles B *a c*, D *f e* are equal and similar in every respect.

30.

In desiring to make the triangle F D E equal to the triangle A B C it may happen that only one of the sides can be measured, B C for instance: then we have recourse to the angles A B C and A C B. Having made D E equal to B C, the lines F D and F E are drawn in such a way that they make with D E the same angles that A B and A C make with B C. Then, by the meeting of these lines is produced the triangle F D E, equal and similar to the triangle A B C. The principle that is assumed in this operation is so simple that it has no need of demonstration.

Fig. 35.

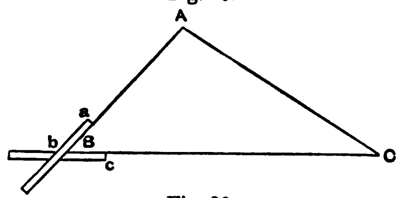
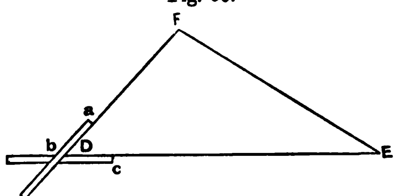


Fig. 36.

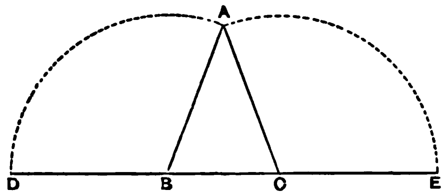


31.

If of three sides of the triangle A B C, one only, the base B C, could be measured, but if this triangle were known to be isosceles, that is to say having the two sides A B and A C equal, it would suffice to measure only one of the two angles A B C and A C B, the other being equal to it.

The reason of this is readily seen on supposing what would happen if the two sides A B, A C, of the triangle A B C, lay in the first place upon B D, and upon C E, prolongations of the base B C, and if then they were turned upwards to meet at the point A. The equality of these two sides would prevent the one from moving farther than the other, and consequently when they are joined, they would be equally inclined to the base B C, or the angle A B C would be equal to the angle A C B.

Fig. 37.



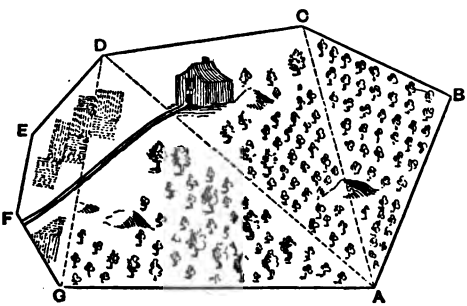
32.

To return to the measurement of lands, it is seen that whatever may be the obstacles presented in their interior, it will be easy, by the preceding method, to transfer to clear land all the triangles dividing the space to be measured.

Suppose, for instance, that we desire to measure a wood, of a figure such as A B C D E F G.

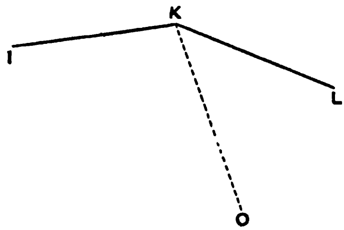
We might first make a triangle equal to A B C, which we could do without entering the interior of this triangle by measuring the two sides A B, B C, and the included angle C B A. This triangle being described would give the angle B C A and the

Fig. 38.



length of C A, and as we could measure the exterior side D C, we should have in the triangle C A D the sides D C and C A. We could find the angle D C A by taking first the angle I K L equal to D C B, and then the angle L K O equal to B C A, which would give the remaining angle I K O equal to the required angle D C A.

Fig. 39.



The triangle A D C thus determined by the two sides D C and C A, and by the included angle D C A, we should know in like manner the triangle D A G and the rest of the figure.

33.

The measurement of lands in which lines cannot be drawn often gives rise to great difficulties in practice. A free uninterrupted space is seldom found sufficiently extensive for making triangles equal to those of the land to be measured; and even if it were found, the great length of the sides of the triangles

Fig. 40.

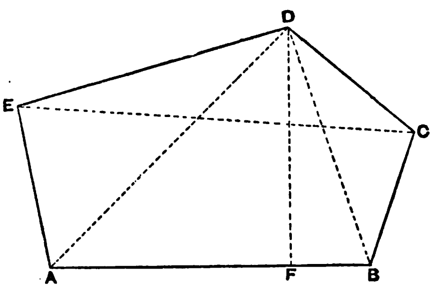
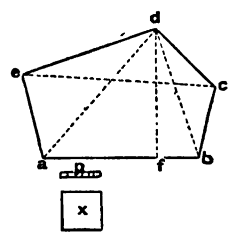


Fig. 41.



might render the operations very difficult. To let fall a perpendicular on a line from a point distant from it only 1000 yards would be extremely difficult, and perhaps impracticable. It is therefore necessary to have a method which takes the place of these large operations. Such a method readily offers itself. It suggests itself that the figure to be measured A B C D E may be represented by a smaller figure *abcde*, in which, for instance, the side *a b* may be 100 inches if the side A B is 100 yards, the side *b c* 45 inches if B C is 45 yards. Then the area of the reduced figure *a b c d e* being 60,000 square inches, that of the figure A B C D E must obviously be 60,000 square yards.

But in the first place we must understand what constitutes the similarity of two figures.

34.

After a little reflection, we must see that the two figures A B C D E, *a b c d e*, in order to be similar, must be such that the angles A, B, C, D, E, of the greater are equal respectively to the angles *a*, *b*, *c*, *d*, *e*, of the less, and further, that the sides *a b*, *b c*, *c d*, &c., of the less contain as many parts such as *p*, as the sides A B, B C, C D, &c., of the greater, contain parts such as P.

35.

In expressing this second condition, Geometers say the sides A B, B C, C D, &c., are proportional to the sides *a b*, *b c*, *c d*, &c., or the side A B contains *a b* in the same manner that B C contains *b c*, or the side A B is as great relatively to *a b* as B C is relatively to *b c*, &c.; or there is the same ratio between A B and *a b*, as between B C and *b c*, or finally, A B is to *ab*, as B C to *b c*, &c. These are all various ways of expressing the same thing, with which one should be familiar, in order to understand the language of Geometers.

36.

Having seen what constitutes the similarity of two figures, let us ask what is the most natural way of drawing one figure similar to another. For this purpose let us imagine that a draughtsman desires to copy a figure to a reduced scale.

First, taking *a b* to represent the base A B of the figure to be copied, A B C D E, he draws the sides *a e* and *b c* inclined in the same way to *a b* that A E and B C are inclined to A B, in observing that the lengths of *a e* and *b c* are to that of *a b*, as the lengths of A E and of B C are to that of A B; that is to say that if A E, for example, be the half of A B, he makes *a e* half of *a b*, and does the same in determining the length of *b c* relatively to B C.

Having thus determined the points *e* and *c*, he draws the two lines *e d* and *c d* inclined to *e a* and *c b* in the same manner as E D and C D are inclined to E A, C B; and prolonging these lines till they meet in *d*, he completes his figure *a b c d e*.

37.

On reflecting on this construction, it is at once seen to depend upon the equality of the angles E, A, B, C to *e*, *a*, *b*, *c*, respectively, and upon the proportionality of the sides E A, A B, and B C to the sides *e a*, *a b*, *b c*, the figure being completed without making the angle *d* equal to the angle D, or the sides *e d*, *c d*, proportional to the sides E D, C D. This reflection might at first raise a doubt whether the angle *d* be actually equal to the angle D, or the sides *e d*, *c d*, proportional to the sides E D, C D, and consequently whether the figure *a b c d e* be altogether similar to the figure A B C D E. Experience alone would soon relieve that doubt; besides, with a little attention it is seen that from the equality of the four angles E, A, B, C, respectively to *e*, *a*, *b*, *c*, and the proportionality of the three sides E A, A B, B C, to *e a*, *a b*, *b c*, necessarily result the equality of the angles D, *d*, and the proportionality of the sides E D, C D, to *e d*, *c d*.

But to avert all possible doubt, let us show that all the conditions required for the similarity of the two figures are necessarily dependent upon one another; which it will be easy to do by examining, first triangles, which are the simplest figures, and which necessarily enter into the composition of all others—an examination which will lead us to all the properties and uses of similar figures.

38.

Let us suppose that upon the base *a b* is drawn the triangle *a b c*, taking only the angles *c a b*, *c b a*, equal to the angles C A B, C B A, of the triangle A B C: it is first to be shown that the third angle *a c b* equals the third angle A C B.

For if the triangle *a b c* be placed upon the triangle A B C in such manner that the point *a* is upon the point A, *a b* upon A B, and *a c* upon A C, it is clear that *c b* must be parallel to C B

Fig. 42.

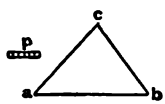
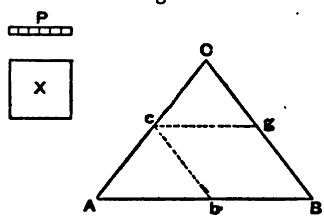


Fig. 43.

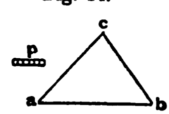


because the side *c b*, prolonged, could not meet the side C B, otherwise the two lines would be unequally inclined to A B, and consequently, the angles *c b* A and C B A would be unequal, which is against the supposition. As from the equality of the angles *c b a* and C B A, it follows that the lines *c b*, C B, are parallels, so from the parallelism of these lines it follows that the angles A *c b*, A C B, must be equal: which was to be proved.

39.

Now let us show that the corresponding sides in the two triangles *a c b* and A C B, including equal angles, are proportional. To fix our ideas, let us suppose first that *a b* is half of A B; then we have to prove that *a c* is likewise half of A C. Let *a c b* (as in the preceding paragraph) have still the position A *c b*, drawing *c g* parallel to A B: this line must equal *b* B or A *b*, and *g* B must in like manner equal *c b*.

Fig. 44.



Then as the angles C *g c* and C *c g* are manifestly equal to the angles *c b* A and *c* A *b*, the triangles C *c g* must equal the triangle *c* A *b* (Par. 30). Hence we have C *c* equal to A *c* and C *g* equal to *c b* or *g* B; therefore A *c* or *a c* is the half of A C and *c b* the half of C B.

If *a b* were contained three, four, or any other number of times in A B, it would be equally easy to show that *a c* would be contained the same number of times in A C, and *c b* in C B. For, from the points of division *b*, *f*, of the base A B drawing *b c*, *f h*, &c., parallel to B C, we could place along A C three, four, &c., triangles A *c b*, *c h g*, *h* C *i*, &c., each equal to the triangles *a c b*, A *c b*.

But if *a b*, instead of being contained an exact number of times in A B, were so contained with some fraction, two and a half times, for example, it could be proved that *a c* would be also contained two and a half times in A C, and *b c* two and a half times in B C.

Fig. 45.

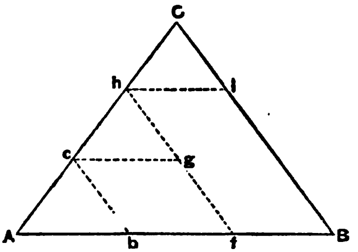
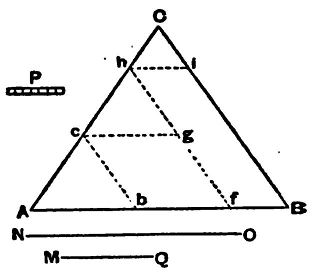


Fig. 46.



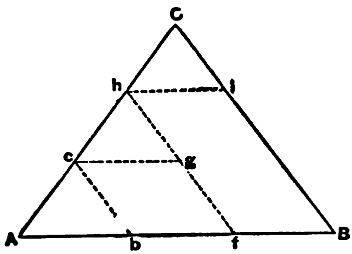
For when by means of the parallels *b c*, *f h*, we should have placed along A C the two triangles A *c b*, *c h g*, equal to *a c b*, there would remain between the two parallels *h f* and C B room to place a triangle C *h i*, of which the sides are halves of the sides of *c* A *b*; which is obvious, because by the supposition *f* B is half of A *b* and the base *h i* of the triangle C *h i* equals *f* B, *h f* being parallel to C B. Therefore, generally, when two triangles A B C, *a b c*, have equal angles, these triangles, named ***similar triangles***, have their sides proportional, or, what is absolutely the same thing, the sides A B, B C, A C, of the one triangle A B C contain the same number of parts P as the sides *a b*, *b c*, *a c*, of the other triangle *a b c* contain of parts *p*, P being the foot, yard, &c., or, generally, the scale on which A B C is constructed, and *p* that employed in constructing *a b c*.

40.

From the proposition just demonstrated flows naturally the solution of a problem often useful in practice.

It is desired to divide a line into a given number of equal parts. This might be done by trial, but never with that certainty given by geometrical exactitude.

Fig. 47.

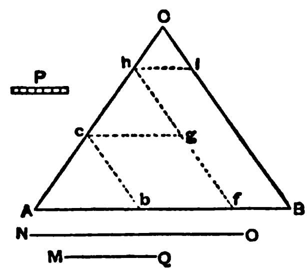


Suppose, for instance, that we have to divide A B into three equal parts, we begin by drawing an indefinite line A C making any angle with A B; then in this line, we set off three equal parts A c, c h, h C, with an opening of the compass taken at will; we then join C B and draw to this line the parallels c b, h f: thus A B cut at the points b, f, is divided into three equal parts; as is obvious from the preceding paragraph.

41.

If it were desired to divide a line into a fractional number of parts, as two and a half, three and a quarter, &c., or, generally, to divide the line A B at the point b so that A B might be to A b as the line N O is to the line M Q, it is seen that the solution of the problem depends on Par. 39, that is to say, we should draw through A any straight line, take upon this line A c and A C respectively equal to M Q and N O, and then draw c b parallel to C B, giving the point b required. Geometers express this problem as follows :—To find a fourth proportional to three given lines N O, M Q, A B.

Fig. 48.



42.

It is evident that two similar triangles A B C, *a b c*, have not only their sides proportional, but also that the perpendiculars C F,

Fig. 49.

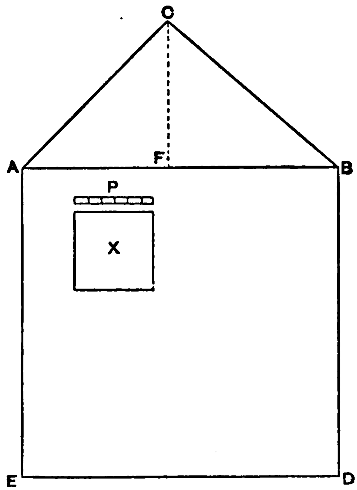
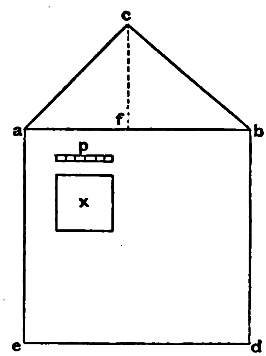


Fig. 50.



c f, if drawn from their vertices C, *c*, upon their bases A B, *a b*, must follow the proportion of the sides. It is so easy to demonstrate this by what precedes, that we need not stop to do so.

43.

As to the area of similar triangles A B C, *a b c*, it is obvious that the area of the former must contain as many squares X made to the scale P, as the area of the second contains squares *x* made to the scale *p*. For as C F and A B have, by the preceding paragraph, as many parts P as *c f* and *a b* have parts *p*, half the product of C F by A B which measures A B C (Par. 14) gives the same number as that which results from half the product of *c f* by *a b* which measures *a b c*, but with this difference, that as C F and A B are reckoned in parts P, their product is counted in squares X, and that as *c f* and *a b* are reckoned in parts *p*, their product is counted in squares *x*.

44.

What has just been said as to the measure of similar triangles serves to prove a proposition which, in the Elements of Geometry, is usually enunciated as follows:—Similar triangles A B C, *a b c*, are to each other as the squares A B D E, *a b d e*, of their homologous, or corresponding, sides A B, *a b*. The demonstration contained in the preceding paragraph leads absolutely to this consequence; for the square A B D E, containing as many X's as *a b d e* contains *x*'s, it is obvious, that the two numbers of square X's which express the relation of the triangle A B C to the square A B D E, are the same as the two numbers of square *x*'s, which give the relation of the triangle *a b c* to the square *a b d e*, or, what is the same, that the triangle A B C is to the square A B D E as the triangle *a b c* is to the square *a b d e*.

Hence it follows that if, for instance, the side A B were double the side *a b*, the triangle A C B would be the quadruple of the triangle *a c b*; that if A B was the triple of *a b*, the triangle A C B would be nine times the triangle *a c b*, &c., for A B cannot be the double of *a b* without the square A B D E being quadruple of the square *a b d e*, &c.

45.

To pass at once from triangles to other figures, let us suppose that to each of the similar triangles A B D, *a b d*, we join two

Fig. 51.

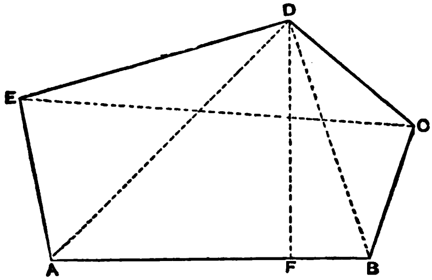
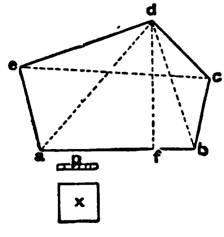


Fig. 52.



other triangles A D E and B D C, *a d e* and *b d c*, respectively similar, we see that in the total figures A B C D E, *a b c d e*:—

(1) The angles A, B, C, D, E, are the same as the angles *a*, *b*, *c*, *d*, *e*, for they are either the corresponding angles of similar triangles, or are composed of such corresponding angles.

(2) The ratios of the homologous or corresponding sides D E, *d e*, B C, *b c*, &c., of the figures A B C D E, *a b c d e*, are necessarily the same, that is to say, if P, for example, is contained a certain number of times in the base A B, and *p* is contained the same number of times in *a b*, P and *p* are contained the same number of times in any two corresponding sides such as D E, *d e*. For owing to the similarity of the triangles A B D, *a b d*, the number of P's contained in A D must equal the number of *p*'s contained in *a d*; therefore, regarding these sides as the bases of similar triangles A D E, *a d e*, the number of parts P contained in D E must be the same as the number of parts *p* contained in the side *d e*.

(3) If in the two figures, corresponding lines be drawn such as C E, *c e*, or the perpendiculars D F, *d f*, &c., these lines must always have the same ratio as the corresponding sides of the two figures A B C D E, *a b c d e*. Thus the figures are precisely similar in all respects.

46.

The figure *a b c d e* being thus described perfectly similar to the figure A B C D E, it is obvious that if it were desired to draw another figure entirely equal to *a b c d e*, and consequently also similar to A B C D E, it would be unnecessary to measure all the sides and all the angles of *a b c d e*; it would suffice, for instance, to take the three sides *a b*, *e a*, *b c*, and the four angles *e*, *a*, *b*, *c*, and with this alone we should be sure of drawing again the same figure *a b c d e* similar to A B C D E. This forms a complete demonstration of that which was previously assumed (Par. 37). But we may go further, for obviously there are different ways of combining the sets of angles and lines that should be measured in any figure, in order to make another figure proportional to it. Further detail would only fatigue the reader.

47.

It might be demonstrated by reasoning similar to that of

Par. 43, that the number of squares X contained in the figure A B C D E, is the same as that of the squares x contained in the figure $a\,b\,c\,d\,e$, and consequently, the areas of similar figures are to each other as the squares of their homologous sides.

48.

All that has been said as to similar figures may be reduced to this single principle, that similar figures differ only in the scales on which they are constructed.

49.

Now to understand better the use that can be made of similar triangles and reductions in measuring land upon which we could not conveniently operate, let us suppose that A B C D E F represents the outline of a part of a park, of a pond, &c., of which we wish to determine the extent. First we should measure one of the sides of the figure, F E for example, and ascertain how many yards, feet, &c., this side contains; then, taking any desired

Fig. 53.

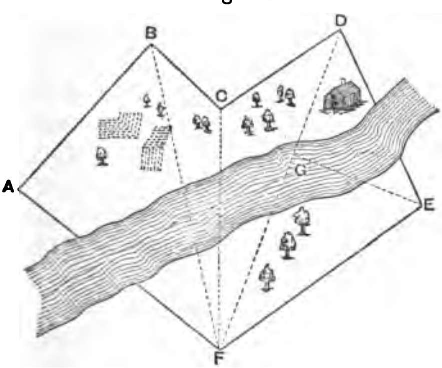
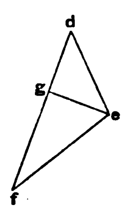


Fig. 54.



scale, we should draw on paper, or cardboard, a line $f\,e$, containing as many parts of the scale as F E contains yards, feet, &c. Then marking the angles $d\,e\,f$, $d\,f\,e$, equal to the angles D E F, D F E, we should have the triangle $e\,d\,f$, in which we should draw *e g* perpendicular to *d f*. This done, and the lines *d f* and *e g* being measured by means of the scale, we should conclude that whatever number these lines contain of reduced parts, so many yards, feet, &c., are contained in D F and E G. Then multiplying D F by the half of E G, we should have the area of the triangle E D F, and measuring in the same way each of the other triangles D C F, B C F, A B F, the area of the whole figure would be determined.

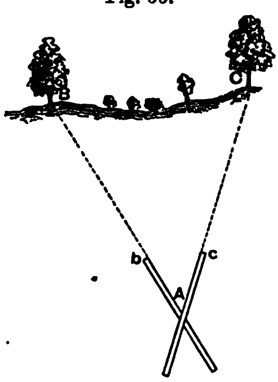
50.

It often happens in practice that it is necessary to measure the distance from the place of observation F, to another place which some obstacle renders inaccessible—a new problem, the solution of which is already given in the preceding paragraph; for, since to measure D F we need only the similarity of the triangles *d e f* and D E F, it is obvious that if we measure any base whatever E F and can from the points F and E see the point D, the problem will be solved, that is to say, we shall have the distance F D.

51.

The use that can be made of special instruments, such as *b* A *c*, (described in Par. 28,) as composed of two branches jointed to the point A round which they can be turned, exposes us to many mistakes. Sometimes the opening of the angle will alter in transport; sometimes the form necessarily given to the instrument to render its use easy will prevent it from being used on the sheet on which the reduction is to be made. Moreover, for each new angle B A C, taken in this way, the instrument has to be placed upon the paper, and the only means of comparing two angles is to place the one upon the other, without

Fig. 55.

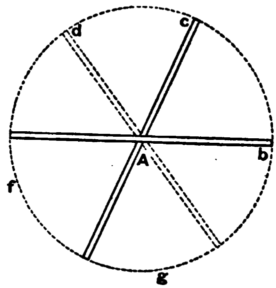


by this means ascertaining precisely, either their relative, or their absolute magnitude.

52.

It was therefore necessary to seek a standard measure for angles, as had been already done for lengths. This was not difficult; for, supposing A *b* to remain fixed, if we joint to it the limb A *c*, and make it turn round A, we can fit to the end *c* of the movable limb A *c* a pen or a pencil, to render the trace of the point *c* visible; this trace, which forms an arc of a circle, will give exactly the measure of the angle for each particular opening of the limbs A *b*, A *c*, that is to say, because of the uniformity of the curvature of the circle, an opening double, triple, or quadruple that of *c* A *b* must have corresponding to it an arc double, triple, or quadruple that of *c b*.

Fig. 56.



53.

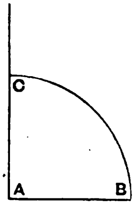
Supposing then that the circumference *b c d f g*, described by the complete revolution of the point *c*, is divided into any number of equal parts, the number of such parts contained in the arc intercepted by the lines A *c* and A *b* will measure the opening of those lines, or the angle *c* A *b* formed by them.

Geometers have agreed to divide the circle into 360 parts, which they call degrees, each degree into 60 minutes, each minute into 60 seconds, &c. Thus, an angle *b* A *c*, for instance, may have 70 degrees, 20 minutes if the arc *b c*, that serves to measure it, has 70 of the 360 parts of the circle, and also 20 sixtieth parts of one degree.

54.

Hence it follows that an angle C A B, of 90 degrees, named

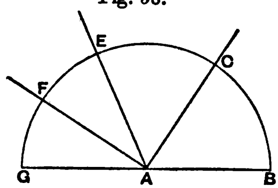
Fig. 57.



commonly a *right angle*, is that of which the sides A C and A B intercept the fourth part B C of the circumference, and are perpendicular to each other.

55.

Fig. 58.



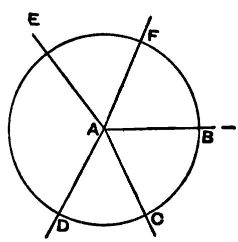
An *acute angle* is any angle smaller than a right angle, or which contains less than 90 degrees. Such are the angles C A B, F A G, E A G.

56.

On the contrary, an *obtuse angle* is that which contains more than 90 degrees, like F A B.

57.

Fig. 59.



It is obvious that all the angles, like G A F, F A E, E A C, C A B, that can be made on the same side of the straight line G B, at the same point or vertex A, are, taken together, equal to 180 degrees, or two right angles, measured by the half-circumference.

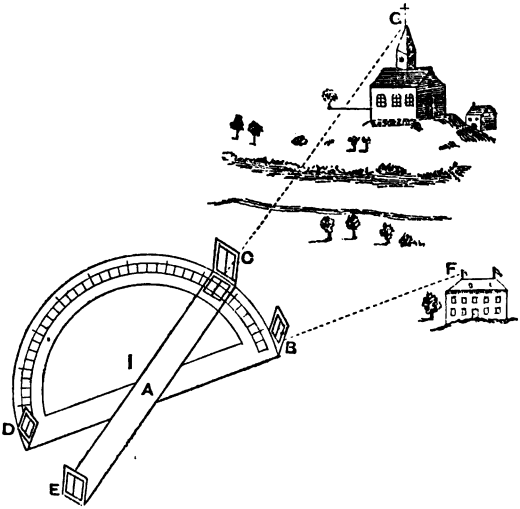
58.

In like manner, the sum of all the angles E A F, F A B, B A C, C A D, D A E, that can be made round the point A, as their common vertex, is equal to 360 degrees, or to four right angles, measured by the entire circumference B C D E F.

59.

After having found that angles are measured by parts of a circle, let us see how to determine how many degrees are contained in an angle, which it is desired to measure. An instrument called a *semi-circle* can be employed for this purpose. It consists of two bars E A C, D A B, of equal length, crossing each other at

Fig. 60.



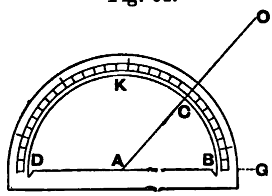
A, and fitted with sight wires at their ends. One of these bars, E C, is movable around A, and the other, D B, is fixed, and forms the diameter of a half-circle D C B, divided into 180 degrees, &c.

Now if we wish to know the angle formed by two straight lines drawn from the place where we stand to any two objects F, G, we first place the fixed bar D A B in such line that the eye, placed at D, can perceive one of the two objects F, in line with the two wires D and B; then, without moving the instrument, we turn the other bar E C so that the eye, placed at E, perceives the other object G, by the wires E and C. The bar E C then marks upon the graduated semicircle the number of degrees, minutes, &c., contained in the proposed angle G A F.

60.

If we wish to make on paper an angle of a certain number of degrees, we employ an instrument K, divided into 180 degrees, which is called a *protractor*. Placing the centre A at the vertex of the angle, and the line A B upon the line A G which is to be one of the sides of the angle, we observe and mark the point C which corresponds to the number of degrees desired in the proposed angle; then through this point and the centre A we draw the line A C O, giving the angle O A G which contains the number of degrees required.

Fig. 61.



61.

Let us suppose now that having taken a base F G upon paper, we wish to make upon this base a triangle F G H similar

Fig. 62.

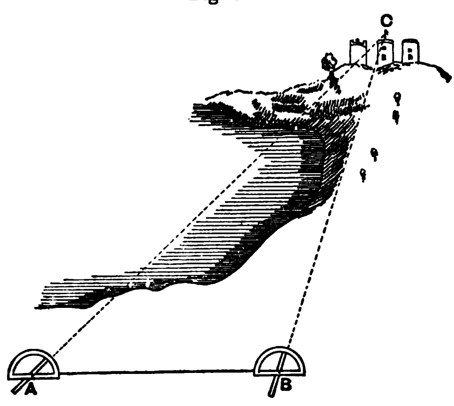
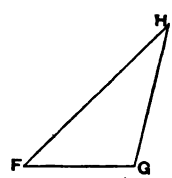


Fig. 63.



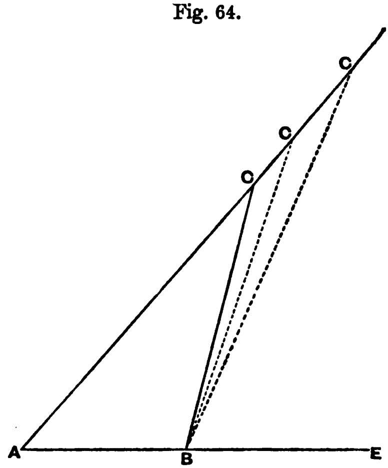
to the triangle A B C taken on the land. We employ the semicircle to ascertain the number of degrees in the angles C A B, C B A, then, by means of the protractor, we make the angles

H F G and H G F, respectively, equal to the angles C A B and C B A, and inasmuch as the point H, where the sides F H and G H meet, is necessarily determined by this operation, as well as the angle F H G, we have the triangle F G H exactly similar to the triangle A B C.

62.

As it is important in practice, as already said, that angles should be exactly measured, it is not enough to observe them, even with the most perfect instruments: means must be found to verify their measure, to correct it, if necessary. There are simple and easy methods of doing so. Returning to the triangle A B C, we feel that the magnitude of the angle C must depend on the magnitude of the angles A and B, for if these angles were increased or diminished, the position of the lines C A, B C, would be changed, and consequently the angle C contained by those lines. If then that angle depends on the angles A and B, we may assume that the number of degrees contained in the angles A and B should determine the number of degrees contained in the angle C, and this should serve to verify the operation performed in determining the angles A and B; for we may be sure that we have correctly measured the angles A and B, if, in afterwards measuring the angle C, we find in it the number of degrees which it should have relatively to the angles A and B.

Fig. 64.

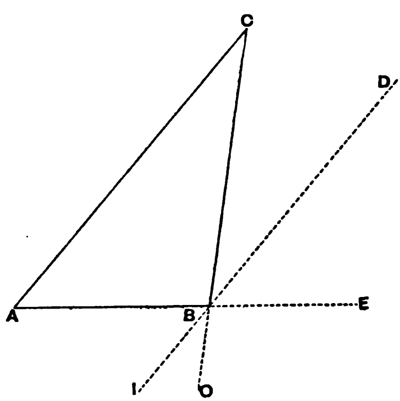


To find how from the angles A and B we can determine the angle C, let us examine what would happen to this angle, if the lines A C, B C, either approached towards, or receded from, each other. Let us suppose, for instance, that B C, turning round the point B, recedes from A B, approaching B E, it is clear that while B C turns, the angle B becomes opened continually, and that on the contrary the angle C becomes more and more contracted. This at first sight would lead us to assume that the diminution of the angle C would equal the augmentation of the angle B, and that therefore the sum of the three angles A B C would be always the same, whatever might be the inclination of the lines A B, B C, to the line A E.

63.

Now, this assumed inference carries with it its own demonstration, for if we draw I D parallel to A C, we see first that the angles A C B and C B D, called *alternate angles*, are equal : which

Fig. 65.



is evident, seeing that as the lines A C and I B are parallel, they must be equally inclined to C B O, and therefore the angle I B O must equal also the angle A C B. But the angle I B O is also equal to the angle C B D, because the line I D cannot be more inclined to C O on the one side than on the other, therefore the angle D B C, equal to the angle I B O, is equal to the angle A B C, its alternate angle.

64.

We see, in the second place, that the angle C A E is equal to the angle D B E because of the parallels C A and D B. Therefore the three angles of the triangle could be put side by side with a common vertex at the point B, and it is seen that the three angles D B E, C B D, and C B A, which are respectively equal to the three angles C A B, A C B, and C B A, are together equal to two right angles (Par. 57).

As all that has been said applies to any triangle whatever, we are assured of this general property, that the sum of the three angles of a triangle is constantly the same, and that it is equal to that of two right angles, or, what comes to the same thing, to 180 degrees.

65.

Therefore, to determine the value of the third angle of a triangle, when we have measured two, we subtract from 180 degrees the number of degrees contained in the two angles together, a property which gives a very convenient mode of verifying the measurement of the angles of a triangle, and which serves many other purposes, as will be seen in the sequel.

We may satisfy ourselves here by deducing the more immediate consequences of this law.

66.

A triangle cannot have more than one right angle; and it is still more obvious that it cannot have more than one obtuse angle.

67.

If one of the three angles of a triangle is a right angle, the sum of the two others is always equal to a right angle.

These two propositions are so clear that they have no need of demonstration.

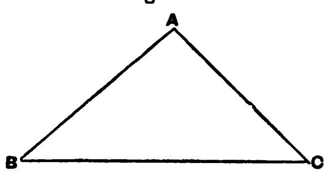
68.

If we prolong one of the sides of the triangle A B C, for instance, the side A B, the exterior angle C B E alone is equal to the two interior opposite angles B C A and C A B, together. For if to the angle C B A be added either the two angles B C A and C A B, or the angle C B E, the sum remains still equal to 180 degrees, or to two right angles (Par. 64).

69.

Knowing one of the angles of an isosceles triangle A B C, we know the two others. Let us have the angle at the vertex A. It is clear that if we subtract the number of degrees contained in this angle from 180 degrees, which is the sum of the three angles of the triangle, half the remainder will measure each of the angles B and C at the base.

Fig. 66.



If one of the two angles B or C at the base were known, the double of its value subtracted from 180 degrees would give the angle A at the vertex.

70.

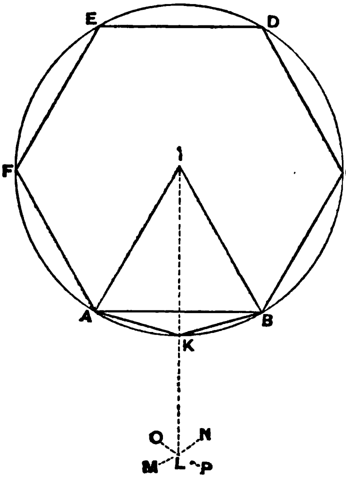
As an equilateral triangle is only an isosceles triangle, of which anyone of the sides may be taken as base, its three angles are necessarily equal, and each contains 60 degrees, the third part of 180 degrees.

71.

Hence is readily deduced the method of describing a hexagon or polygon of six sides (promised in Par. 24).

For, in order to find a line that sets off a sixth part of the circumference, such line must be the chord of an arc of 60 degrees, the sixth part of 360 degrees, or the whole circumference. Let A B be this chord drawing the radii A I and I B, the angle A I B must measure 60 degrees; and, because the two sides A I and I B are equal, the triangle A I B is isosceles. Therefore the angle at the vertex being 60 degrees, each of the two other angles must also measure 60 degrees, the half of 120; so that (Par. 70) the triangle A I B is equilateral, and therefore A B must equal the radius of the circle. Hence it follows that, in order to describe a hexagon, the compass is to be opened to an extent equal to the radius, and to be set six times round the circumference, meeting the six sides of the hexagon.

Fig. 67.



72.

The hexagon A B C D E F being described, the dodecagon, or polygon of 12 sides, can be readily described.

For this, the arc A K B or the angle A I B is to be divided into two equal parts; when A K, chord of the arc A K B, will be one of the sides of the dodecagon.

73.

To divide the arc A K B into two equal arcs, A K, and K B, demands the same operation as to cut the chord A B into two equal parts, that is to say, from the points A and B, as centres, and with any radius, describe arcs M L N, O L P; then, through their intersection L, and the centre I, draw the line L I, which divides into two equal parts both the arc A K B and the chord A B.

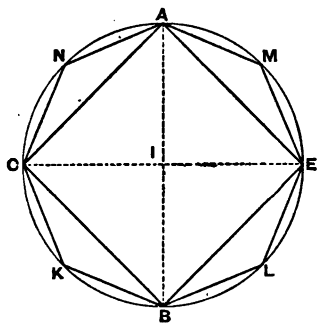
74.

Again by the preceding method dividing the arc A K into two equal arcs, the chord of either is the side of a polygon of 24 sides. In like manner polygons of 48, 96, 192, &c., sides can be described.

75.

To describe an octagon, that is to say, a polygon of 8 sides, we begin by describing a square in a circle—by drawing two diameters A I B and C I E, cutting each other at right angles, and joining their extremities by the lines A C, C B, B E, and A E.

Fig. 68.



For, because of the regularity of the circle and of the equality of the four angles formed by the perpendiculars A I B and C I E, the four sides A C, C B, B E, and E A are necessarily equal and are equally inclined to one another, making a figure which can be no other than a square.

The square being thus described, we can, by the preceding method, divide each of the arcs C K B, B L E, &c., into two equal parts, giving the octagon C K B L E M A N.

By dividing in like manner each of the arcs C K, K B, &c., into 2, 4, 8, &c., equal parts, we get polygons of 16, 32, 64, &c., sides.

SECOND PART.

OF THE GEOMETRICAL METHOD OF COMPARING RECTILINEAL FIGURES.

In giving attention to what has been said in illustration of the discovery of methods of measuring lands, it must have been noted that the relative positions of the lines furnished matters worthy of attention in themselves, independently of the use to which they might be put in practice, and it may be presumed that these matters induced the early Geometers to push still further their discoveries; for it is not necessity alone that urges research, curiosity is often as strong a motive for prosecuting discovery.

The progress of geometry must also have been in great measure due to natural taste for its rigorous precision, without which the mind is never satisfied.

Moreover, in measuring figures it must have been perceived that, in a vast number of cases, scales and protractors gave only approximate values of lines and angles, and there has therefore been a desire for methods that might make up for the defects of such instruments.

We shall now resume rectilineal figures, but in our operations for discovering their true relations we shall use only the rule and the compass.

It often happens that it is necessary either to collect, into a single figure, several figures similar to it, or to divide one figure into other figures of the same kind: this can be done by operating at first upon rectangles, since all rectilineal figures are only collections of triangles, and every triangle is half of a rectangle, having the same height and the same base.

1.

In order to compare rectangles we must learn how to change any rectangle into another having the same superficies, but a different height, for when two rectangles are changed into two others of the same height, they differ only in their bases. The greater is that which has the greater base, and it contains the smaller in the same manner as its base contains the base of the smaller. This is usually enunciated thus: two rectangles having the same height are in the same ratio as their bases.

2.

To add these two rectangles it is only necessary to place one by the side of the other.

3.

It is not more difficult to cut off the smaller from the greater.

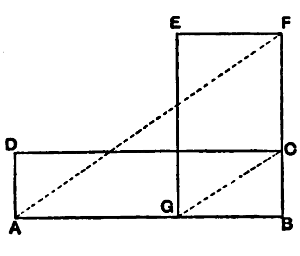
4.

And to divide a rectangle into a certain number of equal rectangles, it is only necessary to divide its base into the like number of equal parts, and draw perpendiculars from the points of division.

5.

Now let it be proposed to change the rectangle ABCD into another BFEG, having the same area but the height BF. It may be remarked that since the area is the product of the height by the base, the rectangle BFEG, having height greater than BC, must have its base smaller than AB, that is to say, if BF for instance be double of BC, BG can only be the half of AB.

Fig. 69.



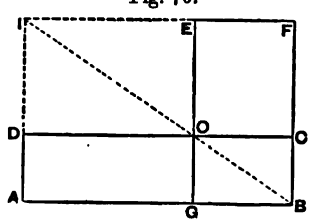
If B F were the triple of B C, B G should be only the third of A B.

In like manner if B F, instead of containing B C an exact number of times, contained it fractionally, as two and a third times, the rectangle B F E G could only be equal to the rectangle A B C D if its base B G were not also contained two and a third times in the base A B. And generally, it will be readily seen that for two rectangles A B C D and B F E G to be equal, the base B G of the one must be contained in the base A B of the other, as the height B C in the height B F. The problem then comes to dividing the line B A in such manner that A B may be to G B as B F is to B C. This is done (1st Part, Par. 41) by drawing the line F A and a parallel C G through C.

6.

To change the rectangle A B C D into another rectangle B F E G, having a given height B F, we may employ a method less natural than the preceding, but more convenient. Having prolonged A D till that it meets in I the straight line F E I, drawn through the point F, parallel to A B, we draw the diagonal B I, and through the point O to where it cuts the side D C, we draw G O E parallel to F B, then the rectangle B F E G will be equal to the rectangle A B C D.

Fig. 70.



To prove this, it will suffice to show that on taking from the rectangles A B C D and B F E G, the common part O C B G, the remaining rectangles A D O G and E O C F are equal.

Now, on noting the equality of the two triangles I B F and I B A, we see that on cutting off from those triangles equal quantities, those remainders must be equal. But the triangle I A B becomes the rectangle A D O G on cutting off the two triangles I D O and O G B, and the triangle I B F becomes the rectangle

E O C F on cutting off the triangles I E O and O B C, equal respectively to the two former: then the two rectangles A D O G and E O C F, remainders of the two triangles, are equal, as well as the rectangles A B C D and B F E G.

7.

This second way of changing one rectangle into another confirms the principle that underlies the first, and which may have seemed to rest only upon a simple inference. From the equality of the two rectangles A B C D and B F E G it would be concluded that A B must be to B G as B F to B C, which is proved by the preceding paragraph. For the triangles I A B and O G B being manifestly similar, the base A B of the greater is to the base G B of the smaller, as the height I A to the height O G, or as B F to B C, their equals. Therefore A B is to G B as B F to B C, conformably to Par. 5.

8.

From this manner of proving that from the equality of the two rectangles A B C D and B F E G, it follows that the height B F is to the height B C as the base A B to the base B C, it may be demonstrated also that when four lines B F, B C, A B, and B G, are such that the first is to the second as the third to the fourth, then the rectangle having for height and base the first and the fourth of these lines, is equal to the rectangle having for height and base the second and the third.

9.

When four quantities, like the preceding lines, B F, B C, A B, and B G, are such that the first is to the second, as the third to the fourth, these four quantities are said to be in proportion, or to form a proportion. Thus 6, 9, 18, and 27 are in proportion, because 6 is contained in 9 in the same manner that 18 is contained in 27. It is the same with 15, 25, 75, and 125, &c.

10.

The first and fourth of the four quantities of a proportion are called *extreme terms*, or simply *extremes*; the second and third are named *mean terms*, or simply *means*. Employing these definitions, the propositions contained in Par. 7 and 8 may be enunciated thus:

11.

When four quantities are in proportion, the product of the extremes is equal to the product of the means.

12.

If four quantities be such that the product of the extremes is equal to the product of the means, then these four quantities are in proportion.

13.

Particular attention should be given to the two preceding paragraphs, as they are of great use. From them is deduced, amongst other things, the demonstration of the rule known in arithmetic as the *Rule of Three*. To give an idea of this rule, let us take an example, as the most simple way of making it understood. Suppose 24 labourers have done 30 yards of work in a certain time, we ask how much 64 labourers will do in the same time.

It is evident that to solve the question we must find a number which is to 64 in the same ratio as 30 is to 24. Now, following what we have seen, this number must be such that its product by 24 shall equal the product of 30 by 64; but the product of 30 by 64 is 1920; therefore the number must be such as being multiplied by 24 will give 1920. Now, however little idea we may have of arithmetical operations, we should easily perceive that this number must be the quotient on the division of 1920 by 24, that is to say, 80.

Generally, to find the fourth term of a proportion of which the three first are given, we take the product of the second and third, and divide this product by the first term of the proportion.

14.

An example so simple as that chosen does not perhaps make the necessity of the preceding method sufficiently felt. Common sense alone might have enabled the required number to be found. Seeing that 30 exceeds 24 by a quarter, and that accordingly the number sought should exceed 64 by a quarter, we get 80. But there are cases where one might have to search for a long time for the relation of the two first numbers of the proportion.

For example, a fourth proportional is required to the three numbers, 259, 407, and 483.

In order to find it by the preceding method, it is necessary to multiply 483 by 407, and to divide 196,581, which is the product, by 259; this gives 759 for the fourth term sought.

If we took other means of finding this term, we could only do it by trial. We might have discovered, for example, that 148, the excess of 407 above 259, contains four of the seventh parts of 259; that, accordingly, we must add to 483 the number 276, which contains four of its seventh parts. But the generality and certainty of the preceding method always saves us from the trouble of making trials which would even in many cases prove fruitless.

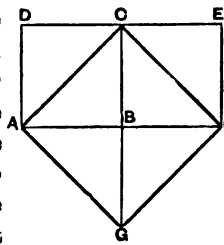
15.

The addition of two squares can be effected in the same manner as that of two rectangles, since the squares are rectangles having their height and the base equal. We should change one of the squares, the smaller for instance, into a rectangle having for its height the side of the greater square, and the two squares will then make one rectangle. We might in like manner give the height of the smaller square to both, or a smaller square at will; but we could not but desire, when we have to reduce two squares to a single figure, some method of making one square equal to two others, a problem of which the following is an easy solution.

16.

Let us suppose in the first place that the two squares A B C D and C B F E, which it is desired to convert into a single square, are equal. It is readily seen that by drawing the diagonals A C and C F, the triangles A B C and C B F are together equal to one square, then transferring below A F the two other triangles D C A and C E F, we form the square A C F G, of which the side A C is the diagonal of the square A B C D and the area of which equals that of the two proposed squares; this needs not to be demonstrated.

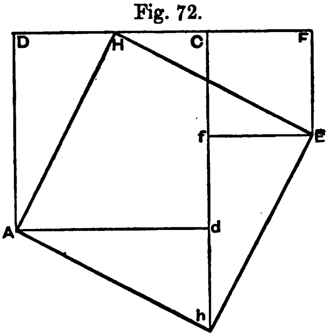
Fig. 71.



17.

Let us suppose now that we wish to make a square equal to the sum of two unequal squares A D C d and C F E f, or, what comes to the same thing, that we propose to change the figure A D F E $f d$ into a square.

Fig. 72.



Following the spirit of the preceding method, we ask if it is not possible to find in the line D F some point H, such,

1. That drawing the lines A H and H E, and causing the triangles A D H and E F H to turn round the points A and E, until they have the positions A d h and E f h these two triangles shall meet in h;

2. That the four sides A H, H E, E *h* and *h* A, shall be equal and perpendicular to each other.

Now this point H will be found by making D H equal to the side C F or E F; for in consequence of the supposed equality between D H and C F, it follows first that if A D H be made to turn round its angle A, so as to take the position A D H, the point H having reached *h*, will be at a distance from the point C, *h* C, equal to D F. From the like supposed equality between D H and C F it follows also that H F is equal to D C, and thus when the triangle E F H is turned round E to the position E *f h*, the point H must reach the same point *h* at a distance from C equal to D F.

Thus the figure A D F E *f d* being changed into a figure of four sides A H E *h*, it only remains to be seen if the four sides of this figure are equal and perpendicular to one another.

The equality of the four sides is manifest, since A *h* and *h* E are the same as A H and H E, and the equality of the two latter results from the fact that D H being equal to C F or to F E, the two triangles A D H and H E F are equal and similar.

It only now remains to be seen if the sides of the figure A H E *h* are at right angles. This is obvious by observing that while H A D turns round A, to reach *h* A *d*, the side H A must make the same movement as the side A D. Thus as the side A D must turn through a right angle D *a d* to become A *d*, the side A H must also turn through a right angle to become A *h*.

As to the other angles H, E, and *h*, they are obviously right angles; for it is impossible that a figure bounded by four equal sides should have one right angle, without the three other angles being also right angles.

18.

On observing that the two squares A D C *d* and C F E *f* are made the one upon A D, the medium side of the triangle A D H, the other upon E F, equal to D H, the smallest side of the same triangle A D H; and that the square A H E *h* equal to the two others is described upon the greatest side A H, which is commonly called the hypothenuse of the right-angled triangle, we readily discover that famous property of right-angled triangles that the square of the hypothenuse is equal to the sum of the squares on the two other sides.

19.

If, then, of two squares, H D L K and A B C D, we desired to make one only, it would be useless to put them at the side of each other and to analyse them as in paragraph 17; it is enough to

Fig. 73.

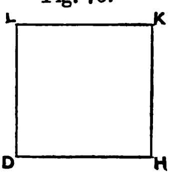


Fig. 74.

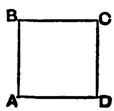
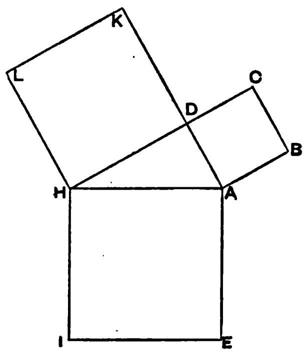


Fig. 75.



place their sides A D and D H so that they may make a right angle and to draw the line A H, which will be the side of the square desired A H I E.

20.

If there are two similar figures D A F G M and D H P O N, and it is desired to construct a third equal in area to the two others taken together, it is only necessary to put the bases A D and H D of these figures upon the two sides of a right angle A D H, and the hypothenuse A H of the triangle A D H will be the base of the figure required.

In order to see the reason of this, let us imagine the squares A B C D and D H K L and A H I E made upon the bases of the three similar figures, then we see, in the first place, by Par. 18, that the one square A H I E is equal to the two other squares A B C D and D H K L. Now, as similar figures are to each other as the squares of their homologous sides (1st Part, Par. 47), the three squares A B C D, D H K L, and A H I E, must be like parts of the figures D A F G M, D H P O N, and A H Q R S.

From this it is readily seen that the figure A H Q R S is equal

Fig. 76.

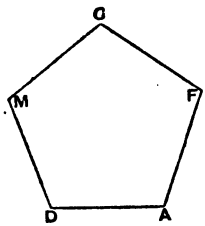


Fig. 77.

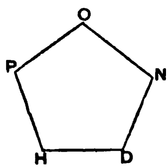
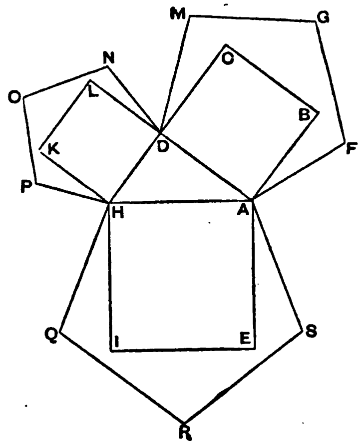


Fig. 78.



to the two others. Let us suppose, for instance, each of these squares to be half of the figure in which it is enclosed, no one would doubt that the figure A H Q R S must be equal to the two others, since its half is equal to the halves of the two figures D H P O N and D A F G M. It would be the same if the squares A B C D, D H K L, and A H I E, were the two-thirds, three-quarters, &c., of the figures D A F G M, D H P O N, and A H Q R S.

21.

If it were proposed to add together three, four, &c., similar figures, or what comes to the same, three, four, &c., squares, the method would remain the same. Let it be desired, for instance, to add three, we should first make a square equal to the two first, then, to this new square we should add the third, and thus we should have one square equal to the three given squares.

22.

Hence it follows that if a square were to be made five, six, &c., times another, it would suffice to follow the preceding method in order to solve this problem; and even its inverse, that is to say to make a square, which should be only the fifth, the sixth, &c., part of a given square; for this it is only necessary to recall the method of finding a fourth proportional to three given lines. In the third Part of this work will be given a method more direct and more convenient for solving problems of this kind.

23.

The addition of similar figures furnishes a decisive proof of the necessity to abandon scales when operations have to be performed in a manner capable of rigorous demonstration.

Let us suppose, for example, that we had to make one square double another; those who did not know the method given in Par. 16 would probably proceed in the following manner.

They would divide the side of the given square into a great number of parts, 100 for example, then multiplying 100 by 100, they would find 10,000 as the value of the square, which would give 20,000 for that of the double square. But from the value of this square, they could not deduce the method of describing it, they would require to have its side expressed by a number, and this number such, that in multiplying it by itself, that is to say squaring it, the product should be 20,000.

But it would be vain to seek this required number upon a scale having parts the hundredths of the side of the first square, for 141, multiplied by itself, gives 19,881, and 142 gives 20,164; both differing (the one less and the other greater) from the square of the required number.

Perhaps they would think that on dividing the side of the given square into more than 100 parts, they might find a certain number of these parts for the side of the square to be double of the first, but whatever attempts they might make, they would find that they sought in vain for two numbers, one of which can express the side, or, using the ordinary terms, the root of the square, and the other, the side or the root of the double square.

24.

Indeed it is demonstrated in arithmetic that if two numbers are not multiples of each other, that is to say if the one does not contain the other an exact number of times, the square of the greater cannot be a multiple of the square of the less. Thus, 5, for instance, which cannot be divided exactly by 4, has for its square 25, which cannot be divided by 16, the square of 4.

If then we square two numbers of which one is greater than the other, but less than its double, there results from this operation two other numbers, of which the one will be less than the quadruple of the other, but neither be its double nor its triple.

If we divide then the side of a square into any number of parts we please, the side of the double square which, according to the demonstration of Par. 16, is the diagonal of this square, cannot contain an exact number of such parts, this is expressed in the language of geometers by saying that the side of the square and its diagonal are incommensurables.

25.

It may be remarked further that there are many other lines which have no common measure. For, when we write the two series

$$\begin{array}{ccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, \&c \\ 1, & 4, & 9, & 16, & 25, & 36, & 49, & 64, & 81, \&c. \end{array}$$

the first expressing the natural numbers and the other their squares, one sees that as the numbers between 4 and 9, between

9 and 16, between 16 and 25, &c., cannot have a root, the sides of two squares of which the one is either triple, or quintuple, or sextuple, &c., of the other, must be incommensurable.

26.

From the fact that many lines are incommensurable with others, perhaps some suspicion might arise as to the exactitude of the propositions that have been used to prove the proportionality of similar figures. In comparing such figures (1st Part, Par. 34 and following), they have always been supposed to have a scale that could be equally applied to measure all their parts, a supposition which as it now seems should be limited on account of what has been said as to incommensurables. We must therefore turn back to examine whether our propositions to be true do not require certain modifications.

27.

Returning in the first place to what was said in Par. 39 of the First Part, let us see if it be exactly true that triangles such

Fig. 79.

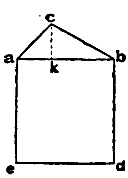
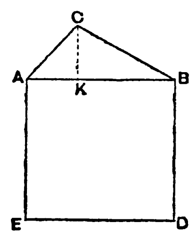


Fig. 80.



as $a\,b\,c$ and A B C, of which the angles are equal, have their sides proportional. Supposing, for example, that the base of the first being $a\,b$, that of the second is a line A B equal to the diagonal of a square of which $a\,b$ is the side, let us see, if, on this supposition, the ratio of A C to $a\,c$ is the same as that of A B to $a\,b$. Although, according to what we have seen, however great may be the number of parts that might be taken arbitrarily in *a b*, A B could never contain an exact number of these parts, it is obvious nevertheless that the greater this number, the nearer does A B approach, being measured exactly by parts of *a b*. If *a b* be divided in 100 parts, A B will contain between 141 and 142 of these parts (Par. 23). Taking 141 as sufficiently near and neglecting the small remainder, it is clear (1st Part, Par. 39) that A C will likewise contain 141 of the parts of *a c*. Suppose now that *a b* is divided into 1000 parts; then A B will contain between 1414 and 1415 of these parts. Taking only 1414 and neglecting the remainder, we shall find that A C contains 1414 of the thousandths parts of *a c*, and that, generally, A C will contain as many parts of *a c* with a remainder, as A B contains of parts of *a b* with a remainder.

Further, these remainders, as we must have remarked, are in each case the smaller the greater the number of parts taken in *a b*, they may therefore be neglected when the division of *a b* is supposed to be carried to infinity: in that case it can be truly said that the number of parts of *a c* contained in A C equals the number of parts of *a b* contained in A B, and that accordingly A C is to *a c* as A B to *a b*.

Thus we have rigorously demonstrated that when two triangles have equal angles, they have their sides proportional, whether those sides have a common measure or not. The proposition (1st Part, Par. 45) from which is deduced the proportionality of corresponding lines in similar figures generally is capable of the same proof.

28.

By like reasoning we see that the propositions explained in Pars. 44 and 47, of the First Part, showing that the areas of triangles and of similar figures are proportional to the squares of their homologous sides, are generally true, even when the sides of these figures are incommensurable.

Let us take, for example, the similar triangles A B C, *a b c*, their heights being supposed to be incommensurable with their bases. In this case there can be no square, however small, for a common measure of those triangles and of the squares made upon their bases, that is to say: the areas *a b c* and *a b d e* are incommensurable, as well as the areas A B C and A B D E; but it is no less true that the triangle A B C is to the square A B D E as the triangle *a b c* to the square *a b d e*.

Of this, we may be assured by observing that the smaller the parts of the scale used in measuring A B and K C are supposed to be, the more nearly do we approximate numbers expressing the relation of A B C to A B D E: therefore, by constantly subdividing the scale of the triangle *a b c*, and neglecting the remainders, the same numbers will always express the ratio of the triangle A B C to the square A B D E, and that of the triangle *a b c* to the square *a b d e*. Imagining the subdivision of the scale to be carried to infinity, the remainders will become absolutely *nil*, and the whole numbers which then express the relation of the triangle *a b c* to the square *a b d e*, express also the relation of the triangle A B C to the square A B D E. Thus the triangle *a b c* is to the square *a b d e* as the triangle A B C to the square A B D E. The same applies to all similar figures.

THIRD PART.

OF THE MEASUREMENT AND PROPERTIES OF CIRCULAR FIGURES.

Having acquired means of measuring all kinds of rectilineal figures, geometers have desired to obtain methods of determining those that are bounded by curved lines. Lands, and generally spaces that are to be measured, are not always bounded by straight lines.

Frequently curvilineal figures and mixed figures, that is to

Fig. 81.

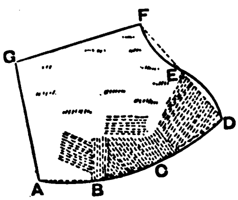


Fig. 82.

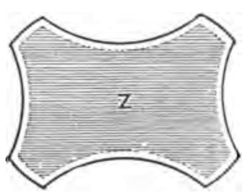
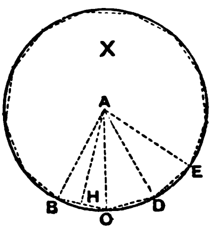


Fig. 83.



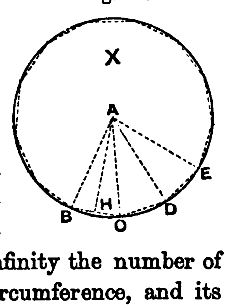
say, those which are bounded partly by straight and partly by curved lines, can be reduced to figures wholly rectilineal, as already stated; for if a figure such as A B C D E F G had to be measured, the side A D might be taken to be a collection of two, three, &c., straight lines. Substituting then the straight line F D for the curve F E D, we should have the rectilineal figure A C B D F G, differing so little from the mixed figure that the one might be taken for the other without sensible error. Such figures could be dealt with by the preceding methods. But geometers would not be content with such operations: they desire only those that are rigorous.

Besides, there are cases where the transformation of a curvilineal figure, or of a mixed figure, into a figure wholly rectilineal, would require the division of its outline into a number of parts so great that the common method would then become impracticable. Nor would anyone be tempted to follow it in measuring a space such as Z or the entire circle X : some other way would have to be taken for measuring spaces of that kind. We shall here discuss only those the outlines of which contain circular arcs.

1.

Suppose, first, that we have to measure the area of the circle X. It will be observed that, by inscribing in it a regular polygon B C D E, &c., the greater the number of sides this polygon has, the more nearly does it approach to the circle. We have seen that the area of a polygon (1st Part, Par. 22) is equal to as many times the product of its side B C by the half of its apothegm A H, as the polygon has sides, or, what comes to the same thing, that this area has for its measure the product of the entire circumference B C D E, &c., by half the apothegm: therefore, since by carrying to infinity the number of the sides of the polygon, its area, its circumference, and its apothegm shall respectively equal the area, the circumference, and the radius of the circle, the measure of the circle is the product of its circumference by half its radius.

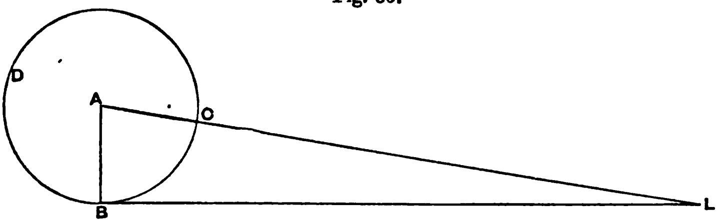
Fig. 84.



2.

It follows that the area of a circle B C D is equal to that of a

Fig. 85.



triangle A B L, of which the height is the radius A B, and the base a straight line B L, equal to the circumference.

3.

It becomes a question then only of having the radius and the circumference. As to the radius, its measurement is easy: not so with the circumference. Yet to obtain its measure, a thread can be wound round the circle, which, in many cases, is near enough for practice. But up to this time no one has succeeded in geometrically measuring the circumference of a circle, that is to say, in determining exactly the ratio that it has to the radius. This ratio can be found to hundred-thousandths, or to millionths, and can be approximated as nearly as desired, yet without being capable of being determined rigorously.

4.

The simplest approximation yet discovered is that ascribed to Archimedes. The diameter being divided into 7 parts, the circumference contains of these parts between 21 and 22, but much more nearly 22 than 21.

5.

It is obvious, however, that if the ratio of any one circumference to its radius were known, we should know that of all other circumferences to their radii, for this ratio should be the same in all circles. This proposition appears so simple that it needs no demonstration, since it is at once perceived that whatever operations are performed in measuring one circumference, using parts of its radius, the same operations would have to be performed in measuring any other circumference, which would necessarily contain the same number of parts of its radius.

6.

Circles have evidently the same general property as all similar figures (1st Part, Par. 47); that is to say, their areas are in the same proportion as the squares of their homologous sides, but as, in applying this proposition to circles, we cannot take

Fig. 86.

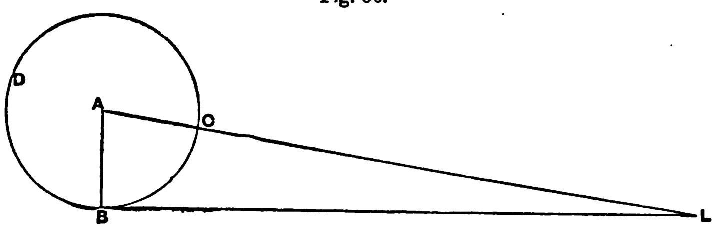
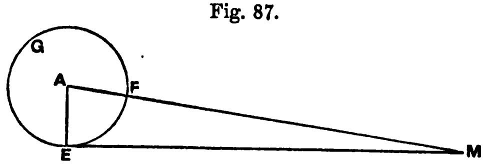


Fig. 87.



their sides, we must use their radii: then it is seen that circles have their areas proportional to the squares of their radii. If it should not appear at first sight that this proposition follows what is said in Par. 47 of the First Part, and if a special demonstration of it should be desired, it is enough to consider that it is the same thing to compare the areas of two circles B C D and E F G, as to compare those of the triangles A B L and A E M, which are respectively equal to them (Par. 2), their bases B L and E M being supposed to be the developments of the circumferences B C D and E F G, and their heights being the radii A B and A E. Now, by the preceding article, these triangles are similar: therefore their areas are in the same proportion as the squares of their homologous sides A B and A E, the radii of the circles B C D and E F G.

7.

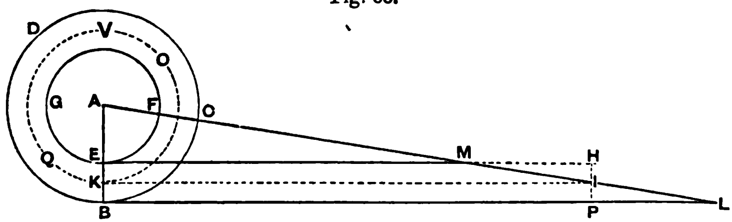
Circles, because of their similarity, have also, like similar figures, this property that if, taking the three sides of a right-angled triangle for radii, we described three circles, that having for its radius the hypothenuse will equal the two others taken together.

Accordingly, a circle can always be found equal to two given circles without taking the trouble to measure each of these circles. Let it be desired, for instance, to make a basin to contain as much water as two others, the depth being the same; or to find the size of a fountain by which will issue as much water as issues by two given pipes, this can be readily done by adopting the method just indicated.

8.

If we had to measure the area of a ring V, a figure contained between two concentric circles E F G and B C D, that is to say,

Fig. 88.



between two circles which have the same centre, it would first suggest itself to measure separately the areas of the two circles, and to subtract the smaller from the larger. But it is easy to discover that the problem can be solved in a way more convenient in practice. Imagine a triangle A B L having the radius A B for its height, the base being a straight line equal to the circumference B C D. Drawing through the point E a straight line E M parallel to B L, this line will equal the circumference E F G; for, the triangles A E M and A B L being similar, there is the same proportion between A B and B L as between A E and E M. Now by the supposition, B L is equal to the circumference of which A B is the radius, therefore, E M is also equal to the circumference having for its radius the line A E, part of A B. The same applies to every other line K I parallel to B L which would measure the circumference of which A K is the radius.

From this equality of the circumference E F G to the line E M follows necessarily the equality of the triangle A E M to the circle E F G; therefore the rectilineal figure E B L M is equal to the ring V. This figure E B L M can be easily changed into a rectangle E B P H by cutting M L into two equal parts M I, and I L, and drawing through the point I the perpendicular H I P, which gives the added triangle M H I, equal to the deducted triangle P L I. Drawing through the point I, I K parallel to B L, as it will cut E B into two equal parts, the proposed ring which is equal to the figure E B L M or E B P H, will have for its measure the product of E B by K I the circumference of which A K is the radius. Therefore, in order to measure a ring V, we multiply its width by the circumference K O Q, which is called the *mean* between the circum ference B C D and E F G, because it exceeds the smaller circumference E F G, or the line E M by a quantity M H, equal to P L, the quantity by which it is exceeded by the greater circumference B C D or the line B L.

9.

When we have to measure a figure Y, composed of arcs of different circles and straight lines, or a figure Z, wholly composed of circular arcs, the whole difficulty reduces itself to the measurement of segments of circles, that is, of such spaces as A B C E, bounded by an arc A B C, and a chord A C; for figures entirely composed of circular arcs, or of arcs and straight lines,

Fig. 89.

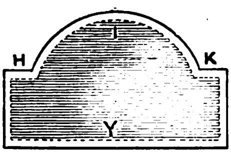
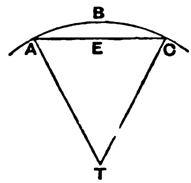


Fig. 90.

Fig. 91.



may all be considered to be rectilineal figures, increased or diminished by certain segments.

10.

The measurement of a segment A B C E is easy when we know that of the circle; for on drawing the lines A T and C T to the centre T of the arc, we form a figure A B C T, called a *sector*, of which the area is to that of the circle as the arc A B C is to the entire circumference, and which, consequently, is measured by the product of half the radius A T by the arc A B C. The sector being thus determined, it is only necessary to subtract from it the triangle A C T to have the segment A B C E.

11.

As it often happens that in measuring a figure such as Y, the centre of the arc H I K is unknown, and yet without this centre the figure cannot be measured, since the preceding method requires the radius to be known, it is necessary to find means of ascertaining the centre of any circular arc.

Let A B C be the given arc. Taking at will two points A and B upon this arc and from the points, as centres, describing the four arcs *g o i*, *f o h*, *l p k*, and *m p n*, the two first with any radius, and the two others either with the same or any other radius, it is clear that the required centre of the arc A B C must be upon the line $o\ p$, joining the points of intersection o and p.

Choosing then a third point C upon the arc A B C, and

Fig. 92.

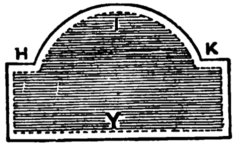
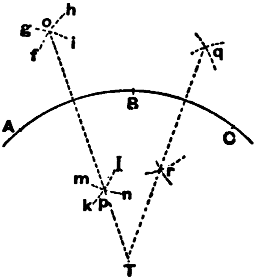


Fig. 93.

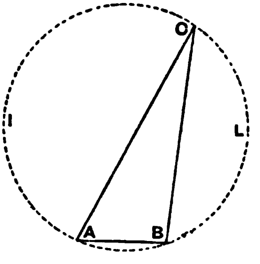


making use of B and of C in the same way as A and B were used, we get a straight line $q\ r$, upon which also the required centre must be: therefore this centre must be the point of intersection T of the lines $o\ p$ and $q\ r$.

12.

Thus, however three points be arranged, provided they are not in one straight line, they can always be connected by an arc of a circle, or, what comes to the same, whatever be the proportion of the sides A C and B C of a triangle A C B to its base, a circle can always be circumscribed about that triangle.

Fig. 94.



13.

The method just given of circumscribing a circle about a triangle being applied successively to different angles A B C, A E B, and A G B, more or less high in relation to their base A B, it is seen that in passing from a triangle A C B, having a very acute angle at the vertex, to other triangles A E B and A G B, having greater angles, the centre of the circumscribed circle continually approaches nearer to A B, and this centre passes afterwards to the under side of A B, when the angle at the vertex A G B has attained a certain magnitude. Now, seeing this centre pass below A B after having seen it above, should, it seems, suggest an inquiry as to what kind of triangle A F B it is that has the centre of the circumscribed circle exactly upon A B.

In ascertaining this triangle A F B we notice, in the first place, that in this particular case, the portion of the circle cir-

Fig. 95.

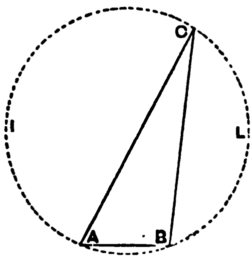


Fig. 96.

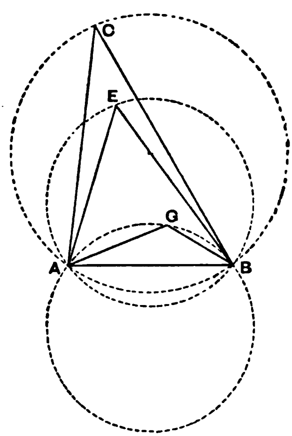
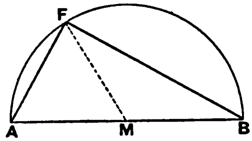


Fig. 97.



cumscribed about the triangle must be exactly a half-circle. Indeed, as the centre of the circle is to be upon the base A B, of which the extremities are, by supposition, on the circumference, the centre M must necessarily be situated precisely at the middle of A B in order that A B may be a diameter. We then see that whatever be the point F of the half circle from which are drawn the lines F A, and F B, the angle A F B must be a right angle; for joining F M the two triangles A F M and M F B are isosceles triangles: therefore the two angles A F M and M F B are respectively equal to the angles F A M and F B M, and consequently the total angle A F B equals the sum of the two angles F A M and F B M; but as the three angles A F B, F A M, and F B M, taken together amount to two right angles, the angle A F B must be a right angle.

Accordingly, if upon the base A B any right-angled triangle whatever be described, this triangle will have the required property of being inscribed in a circle of which the centre is upon the base.

14.

This property of the circle, that the angle in the half-circumference subtended by the diameter is always a right angle, leads us to inquire if the other parts of the circle have some analogous property; if, for instance, the angles A C B, A E B, and

Fig. 98.

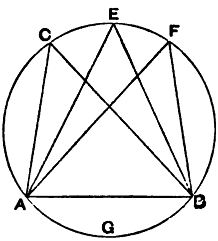
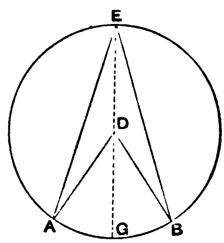


Fig. 99.



A F B, taken in a segment A C E F B, be not all equal as those in the semicircle are.

To make sure of this, we may begin by ascertaining the value of one of these angles, and we can afterwards see if the others are of the same value. We take, for example, the angle A E B, the vertex E of which is at the middle of the arc A E B. As the line E D G, which passes through the centre D, cuts this angle into two equal parts, it is sufficient to measure the half-angle A E G, or, what comes to the same thing, it suffices to learn what part the angle A E G is of an angle already measured, such as A D G. The angle A D G is, as we know, already measured by the arc A G (1st Part, Par. 52).

Noticing that the triangle A E D is isosceles, we readily see that the angle A E G is half the angle A D G; for the angles A E D and E A D (1st Part, Par. 31) are equal. But (1st Part, Par. 67) these two angles, taken together, equal the exterior angle A D G; therefore the angle A E D or A E G is half the angle A D G.

For the same reason, the angle D E B, is half the angle G D B: therefore the whole angle A E B is half of the angle A D B, and its measure is consequently half of the arc A G B.

15.

The angle A E B being measured, in order to ascertain if it is equal to each of the other angles in the same segment, we must examine if some one of these angles, A F B, for instance, is also half of the angle at the centre A D B. This is readily seen by drawing the line F D G through the centre, for then it is obvious that the

Fig. 100.

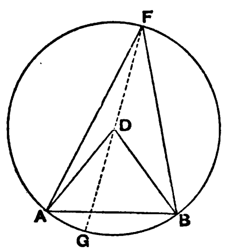
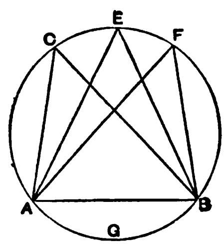


Fig. 101.

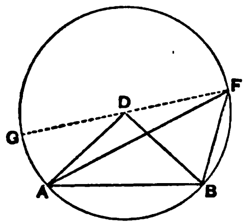


angle A F B is composed of two others, A F D and D F B, which, by the preceding paragraphs, are the halves of the angles A D G and G D B; whence we conclude that the whole angle A F B is half of the angle A D B; and on applying the same reasoning to all the angles A C B, A E B, and A F B, in the same arc A G B, we can show that these angles are equal to each other, as was presumed in the preceding paragraphs.

16.

Among the different angles in the arc A C E F B, there are some which might at first sight appear not to be included in the preceding demonstration: these are angles like A F B such as the line F D G, drawn through the centre, passes outside of the angle A D B. But observing still that the angle G F A is half the angle G D A, half the angle G D B, and the angle D F B, we see that the angle A F B, the excess of the angle D F B above the angle D F A, is, in this case, half the angle A D B which is the excess of the angle G D B above G D A.

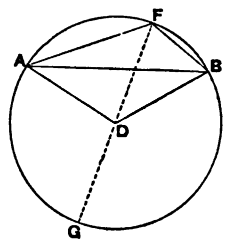
Fig. 102.



17.

From the figures which we have employed, it would seem that the preceding demonstration applies only to segments larger than a semicircle; but it is easy to see that any angle, such as A F B in a segment smaller than a semicircle, always consists of two others, D F B and D F A, halves of the angles B D G and A D G; and consequently that this angle A F B is measured by the half of the two arcs B G and A G, that is to say, the half of the arc A G B.

Fig. 103.



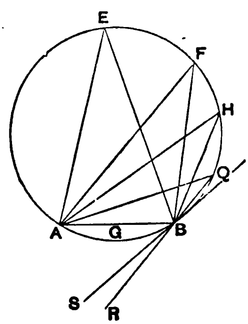
18.

Having seen that in the same segment the angles A E B, A F B, and A H B, at the circumference, are all equal, we are induced to inquire what the angle A Q B becomes when its vertex coincides with the point B, the extremity of the base A B. Would this angle then vanish? It appears not possible that, without being gradually diminished, it should at once come to nothing. We cannot indicate the point beyond which this angle would cease to exist; how then could it be determined? This is a difficulty that cannot be solved without having recourse to the Geometry of the Infinite, of which everyone has at least an imperfect idea which needs only to be developed.

Let us observe in the first place that as the point E approaches B, taking successively the positions F, H, Q, &c., the line E B becomes continually shorter, and that the angle E B A, which it makes with the line A B, becomes greater and greater. But, however short the line Q B may become, the angle Q B A is not the less an angle, since, to render it visible, it needs only to prolong the shortened line Q B towards R.

Should it be the same when the line Q B, in consequence of its diminution, is reduced at last to nothing? What is then its position? What is its prolongation? It is obvious that it is nothing but the line B S, which touches the circle in one point only B, without meeting it in any other part, and which, for this reason, is called the *tangent*.

Fig. 104.



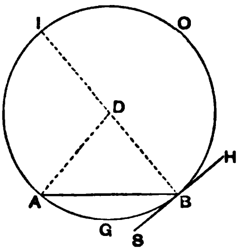
Further, it is obvious that whilst the line E B continually diminishes until it becomes nothing at last, the line A E, which becomes successively A F, A H and A Q, &c., approaches always to A B, and at last coincides with it: then the angle at the circumference A E B, after having become A F B, A H B and A Q B, becomes at last the angle A B S, contained by the chord A B and the tangent B S; and this angle, which is called *the angle in the segment*, should still retain the property of having for its measure the half of the arc A G B.

Although this demonstration may be perhaps a little abstract for beginners, it has been thought right to give it, because it will be very useful for those who may desire to carry their studies on to the Geometry of the Infinite to be accustomed early to considerations of this kind. If, meanwhile, beginners should find this demonstration beyond their powers, it is easy to put it within their reach to discover another by explaining to them the chief property of tangents.

19.

This property is that a tangent to a circle in any point B is perpendicular to the diameter I D B which passes through this point, for as the curvature of the circle is so uniform that any diameter divides it in two half-circles, I A B and I O B, equal and equally situated with respect to this diameter, the two parts B S and B H of the tangent common to these two half-circles must likewise be equally situated with respect to this diameter. This can only be so when I D B is perpendicular to the tangent H B S.

Fig. 105.



20.

This will readily show why the angle in the segment A B S has for its measure half the arc A G B.

For the angle A D B, together with the two equal angles D A B and D B A, makes (1st Part, Par. 64) two right angles; therefore, half the angle A D B, together with the angle D B A, makes one right angle. But the angle D B A added to the angle A B S makes also a right angle; therefore the angle A B S, is half of the angle A D B, and consequently the measure of A B S is half of the arc A G B.

21.

The second demonstration just given of this property of the circle, that the angle A B S is measured by half the arc A G B, furnishes the solution of the following problem.

To describe upon A B a segment of circle to contain a given angle L, that is to say, a segment A F B in which all the angles A F B at the circumference shall be equal to the angle L.

To solve this problem we make at A and at B angles B A S and A B S, each equal to the angle L, and erect from A S and B S the two perpendiculars A D and B D. Their intersection D will be the centre of the required arc A F B.

For, by Par. 19, the lines B S and A S are tangents to the circle of which the centre is D, and the radius A D or B D, since B D or A D are perpendiculars respectively to B S and to A S. Further, by the preceding paragraph, the angle A B S has for its

Fig. 106.

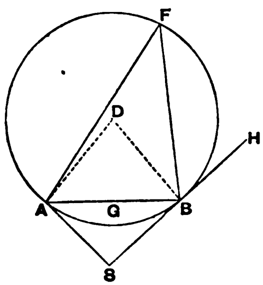
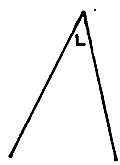


Fig. 107.



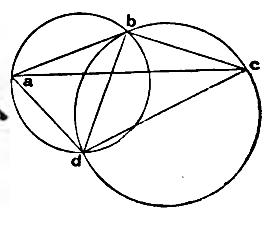
measure the half of A G B, and, by Par. 15, angles such as A F B are also measured by the half of A G B, therefore these angles A F B are equal to A B S, that is to say, to the angle L, as was required.

22.

The discovery of the properties of circular segments that we have just explained was probably due to the mere curiosity

Fig. 108.

Fig. 109.



of geometers; but with this discovery it has been, as with many others: what was not at first believed to be useful became so afterwards. In practice many happy applications have been made of the properties of the circle that we have just demonstrated. A single case may be given in the solution of the following problem which is often necessary in geography.

A, B, and C, are three places of which the respective distances A B, B C, and A C, are known, the question is to ascertain at what distances from these places is a point D, whence the three places can be seen, but are inaccessible for operations on the land. Drawing on paper three points, *a*, *b*, and *c*, situated in relation to one another similarly to the three points A, B, and C, that is to say, in geometrical language, let us make the triangle *a b c* similar to the triangle A B C.

Having then observed with the semicircle the magnitude of the angles A D B and B D C, we draw upon *a b* a segment of a circle *a d b* to contain the angle A D B, and upon *b c* we draw a segment *b d c* to contain the angle B D C. The intersection *d* of these two segments indicates on the paper the position of the place D, that is to say, the lines *d a*, *d b*, and *d c*, bear the same proportion to *a b*, *b c*, and *a c*, as the distances sought D A, D B, and D C, bear to the given distances A B, B C, and A C, which needs not to be demonstrated after what has been already seen as to similar figures.

23.

It might be readily shown that practice has derived much

Fig. 110.

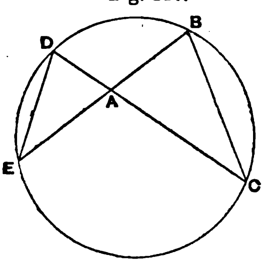
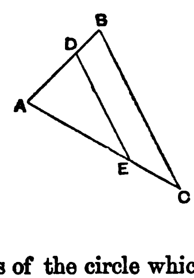


Fig. 111.



aid in other respects from the properties of the circle which we have demonstrated, but it is better to pass to other properties of the circle which have been deduced from the preceding and which have also had their uses.

To proceed regularly to the discovery of these properties, let us first observe that any two angles E D C and E B C, on the same arc E C, being equal, the triangles D A E and B A C have equal angles, that is to say (1st Part, Par. 39), these triangles are similar.

For, by the same reason that the angle E D C is equal to the angle E B C, the angle D E B is equal to the angle D C B, and the angles D A E and B A C are obviously equal, either because they are contained by the same lines, or because two triangles, of which the one has two angles respectively equal to two angles of the other, have also necessarily their third angles equal (1st Part, Par. 38).

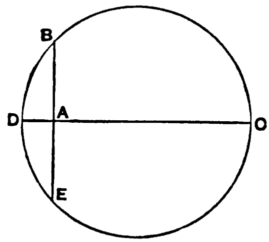
In order to recognise more easily in the triangles A D E and A B C the general properties of similar triangles, we may apply the triangle D A E to triangle B A C by putting A D upon A B, and A E upon A C, so that D E is parallel to B C. We then remember—

1. That if two triangles A D E and A B C are similar, the four sides A C, A E, A B, and A D are proportional (1st Part, Par. 39).

2. That in every proportion the product of the extremes is equal to the product of the means (2nd Part, Par. 8), and we thence conclude, that the rectangle or the product A C by A D is equal to the rectangle of A E by A B, a very remarkable property of the circle, which may be thus enunciated: if in a circle be drawn any two straight lines cutting each other, the product of the two parts of the one is equal to the product of the two parts of the other.

24.

Fig. 112.

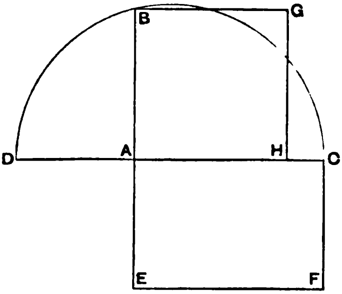


If the two straight lines B E and D C cut each other perpendicularly, and if one of these be a diameter D C, the two parts A B and A E, of the other line B E, are manifestly equal, so that the former property would in this particular case be thus stated: if upon the diameter D C of a circle any perpendicular A B be erected, the square of that perpendicular is equal to the rectangle of A D by A C.

25.

It often happens that a rectangle has to be changed into a square: the preceding paragraph furnishes a ready means of doing so.

Fig. 113.



Let A C F E be the rectangle proposed, prolong A C to D, so that A D may be equal to A E, and describe the semicircle D B C on the diameter D C. Prolonging then the side E A till it meets the semicircle, A B is the side of the required square A B G H, equal to the given rectangle A C F E.

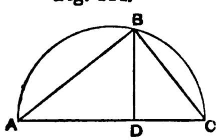
26.

A problem is often proposed which is only that which has just been solved presented in another form, namely, to find a line which shall be a mean proportional between two given lines. By a mean proportional is understood the line which is as much greater relatively to the smaller of the two given lines, as it is smaller relatively to the greater, that is to say if A B, for instance, is a mean proportional between A D and A C, we can say that A D is to A B as A B is to A C. It is easy to see that this problem is the same as the preceding, since (2nd Part, Par. 8) the product of A D by A C, that is to say the rectangle of these two lines, must be equal to the product of A B by A B, that is to say to the square of A B. Thus when a mean proportional between two given lines is to be found, we have only to change the rectangle of these two lines into a square of which the side will be the line required.

27.

A mean proportional between two lines can also be found in another way, which follows from the property of the circle explained in Par. 13. Let A C be the greater of two given lines, and A D the less, we draw D B perpendicular to A C, and the point B, where it meets the semicircle A B C, drawn upon A C as diameter, gives the line A B, a mean proportional between A D and A C. For joining B C it is obvious that the triangle A B C has a right angle at B, therefore (1st Part, Par. 38) that triangle is similar to the triangle A B D, since these two triangles have also the common angle A, but if the triangles A D B and A B C are similar, they have their sides proportional, therefore A D is to A B as A B to A C: thus A B is a mean proportional between A D and A C.

Fig. 114.



28.

In changing any rectilineal figure whatever into a square, it is only necessary, in order to bring this problem under Par. 25, to change this figure into a rectangle, which it is easy to do, because rectilineal figures are only collections of triangles, every triangle is the half of a rectangle on the same base, having the same height, and all the rectangles resulting from the triangles can be collected into one by giving them all the same height (2nd Part, Par. 6).

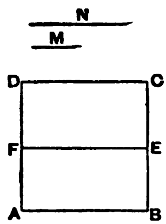
29.

Figures of which the outlines contain arcs of circles can also be changed into squares, when the length of the arcs of which they consist have been practically measured; for these figures, like rectilineal figures, can be changed into rectangles, as pointed out in Pars. 9 and 10, which teach how to measure all kinds of circular figures.

30.

From the property of the circle, explained in Par. 24, is deduced an easy method of making a square which shall be to a given square in a given ratio, a problem promised in Par. 22 of this Second Part.

Fig. 115.



Suppose, for instance, that a square is to be drawn which shall be to the square A B C D as the line M to the line N. We divide (1st Part, Par. 41) the side C B, at the point E, in such manner that C B is to B E as the line N to the line M, and, drawing E F parallel to A B, then, as the rectangle A B E F will have the same area as the square required, it remains only to change the rectangle into a square.

31.

If it were desired to make a polygon H I K L M, which should have to a similar polygon A B C D E the ratio of the line X to the line Y, we may commence by making upon the side A B of the given polygon A B C D E, the square A B G F, then finding another square H I O Q which is to the square A B G F as the line X to the

Fig. 116.

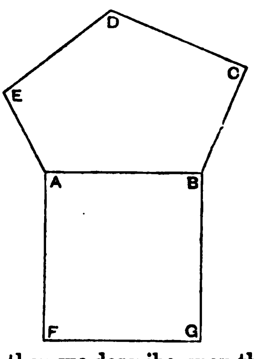
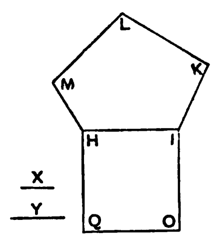


Fig. 117.



line Y; then we describe upon the side H I of this square a polygon H I K L M similar to the first A B C D E, and this new polygon will be that required. The reason of this is obvious, on recalling, 1st Part, Par. 48, that similar figures are to one another as the squares of their homologous sides.

32.

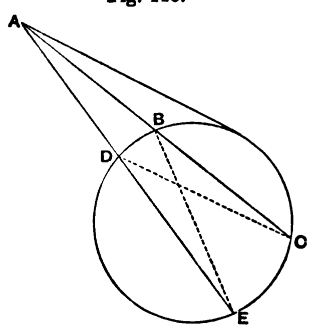
If we desired to make a circle of which the area should be to that of a given circle as X to Y, we should construct a square which should be to the square of the radius of the given circle as X to Y, and the side of this new square would be the radius of the square required.

33.

Another property of the circle is deduced from that which has furnished the preceding problems.

If from a point A, taken outside of a circle, two straight lines, ABC and ADE, are drawn, each cutting the circumference in two points, and if the straight lines CD and BE are drawn, the triangles ACD and AEB are similar, since the angle A is common to both, and they have also the angles at the circumference C and E equal. Now, from the similarity of the triangles ACD and AEB, it follows that the four lines AB, AD, AE, and AC are in proportion, and, consequently, that the rectangle of the two lines AB and AC is equal to the rectangle of the two lines AD and AE, which may be expressed thus: if from any point whatever A, taken outside of a circle, any two straight lines AC and AE be drawn to cut the circle, the rectangle of the one line AC by its exterior part AB is equal to the rectangle of the other line AE by its exterior part AD.

Fig. 118.



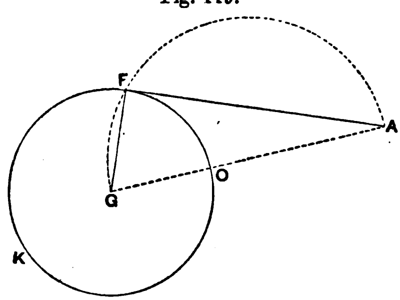
34.

When the line drawn from the point A, instead of cutting the circle, merely touches it, as A F, the preceding property is thus changed: the square of a tangent A F is equal to the rectangle of any secant A E by its exterior part A D. This it is easy to demonstrate; for, regarding the line A F which touches the circle as a line which cuts it at two points infinitely near each other, the lines A B and A C become then the one line A F, and instead of the rectangle of A B by A C, we have the square of A F.

35.

The proposition demonstrated in the preceding paragraph, while it teaches us the value of the square of the tangent A F,

Fig. 119.



does not teach us how to draw this tangent from the given point A.

To draw it, we must remember (Par. 19) that the radius F G is perpendicular to the tangent F A. Accordingly, we have only to find upon the given circle the point F so that the angle A F G may be a right angle; then, by describing upon A G a semicircle, the point where it cuts the circle F K O is (Par. 13) the required point F.

FOURTH PART.

OF THE MEASUREMENT OF SOLIDS AND THEIR SURFACES.

The principles established in the three first Parts of this work might suffice for the solution of problems much more difficult

Fig. 120.

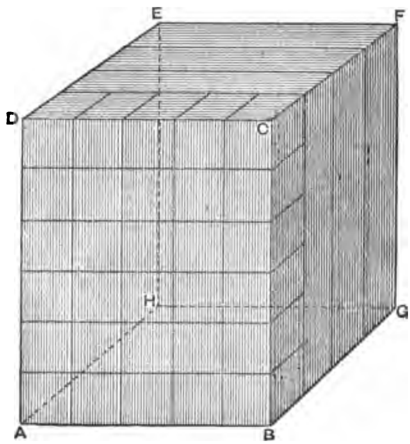
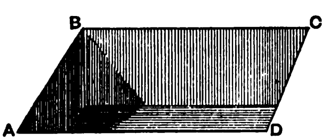


Fig. 121.



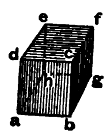
than those about to be proposed, but it is more in the order previously followed to pass now to the measurement of solids, that is to say, of bodies of limited extent, which have at once three dimensions, length, breadth, and thickness. This inquiry has doubtless been one of the first objects to fix the attention of geometers. It may have been desired to ascertain, for example, how much masonry there was in a wall of which the height A D, the breadth A B, and the depth B G were known. It may have been proposed to determine the quantity of water contained in a trench or reservoir such as A B C D; or to find the solidity of a tower, an obelisk, a house or a steeple, &c.

In treating of figures which have three dimensions, in the same manner as we have treated those which have two dimensions only, we will begin by examining the solids which are bounded by planes. We need not discuss methods of measuring the surfaces of such bodies; they can be only assemblages of rectilineal figures, and consequently their measurement depends on what has been said in the First Part.

1.

In measuring the solidity of bodies, it is natural to bring them all into relation with the most simple solid, as in measuring surfaces they have been all related to the square.

Fig. 122.



Now, the simplest solid is the cube, which in reality is among solids what the square is among surfaces; that is to say, it is a space such as *abcdefgh*, of which the length, breadth, and depth are equal, or, what comes to the same thing, it is a figure bounded by six equal faces which are squares.

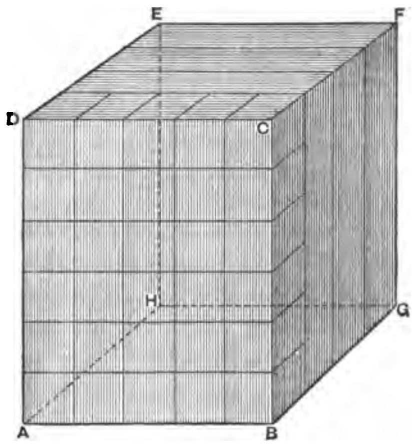
The *side of the cube* is the side of the square which forms its faces.

By a cubic foot is understood a cube of which the side is a foot; in like manner, a cubic inch is a cube of which the side is an inch, &c.

2.

The solids that have generally to be measured are figures such as A B C D E F G H, bounded by six rectangular faces, A B C D, C B G F, C F E D, D E H A, G F E H, and A B G H. These solids are called *parallelopipeds*, because their opposite faces, preserving everywhere the same distance from each other, are called

Fig. 123.



parallels, just as lines are also named parallels, when they preserve always the same distance.

3.

Now, in proposing to measure solids of this kind, the analogy of this problem to that of measuring rectangular surfaces gives a ready means of solving it. In the first place, measuring separately the length A D, the width A B and the depth B G of the proposed figure either in feet or in inches, &c., we multiply by one another the three numbers found, and the product of this multiplication will express how many cubic feet or cubic inches the parallelopiped contains, according as the dimensions are measured in feet or inches. The better to show how this operation is performed, we may take the following example.

Let the length A D be 6 feet, the width A B 5 feet, and the depth B G 4 feet, then the rectangle A B C D (1st Part, Par. 11) will contain 6 times 5 or 30 square feet. If we imagine then that the lines B G, C F, D E, and A H, which all equally measure the depth of the solid, are divided each into four equal parts, and that through the corresponding points of division are drawn as many planes parallel to each other, those planes will divide the proposed parallelopiped into four other parallelopipeds, each one foot in depth and all equal and similar.

Now, simple inspection of the figure shows that the first of these parallelopipeds contains 30 cubic feet, since its exterior face A B C D contained 30 square feet; therefore the whole solid A B C D E F G H will contain 4 times 30, or 120 cubic feet.

4.

We shall not stop to explain the different means that can be used in practice for constructing parallelopipeds, because these means are, for the most part, so easy to find that no one could fail to devise them. But we may give the following mode of forming the parallelopiped, which it is more useful to consider than any of the others.

If we conceive that a square or rectangle A B G H is moved parallel to itself in such a way that its four angles A, B, G, and H, traverse each of the four lines A D, B C, G F, and H E, perpendiculars to the plane of the rectangle A B G H, that rectangle by the movement now described, will form the parallelopiped A B C D E F G H.

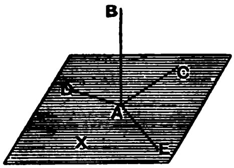
5.

It is hardly necessary to state that by a line perpendicular to a plane we understand a line which does not incline towards any side of that plane; and, in like manner, a plane which does not incline more to one side than to another upon a second plane is said to be perpendicular to that second plane. These two definitions are analogous to that given of a line perpendicular to another line.

6

It follows from this that the line A B, which is perpendicular to the plane X, must be perpendicular to every line A C, A D, A E, &c., in that plane drawn from the foot A of the perpendicular; for it is obvious that if it inclined towards any one of these lines, it would be inclined towards some part of the plane, therefore it would not be perpendicular to it.

Fig. 124.



7.

In order to represent clearly to the mind how the line A B can be perpendicular to all the lines which extend from its extremity A, a figure may be made in relief in the following manner.

Making in some flat flexible substance, like cardboard, a rectangle F G D E divided into two equal parts by the line A B, perpendicular to the sides E D and F G, we can fold that rectangle along the line A B, and place it so folded upon the plane X. It is obvious that, however much or little the two parts F B A E and G B A D of the folded rectangle E A D G B F may be opened apart, those two parts will always be in contact with the plane X, without the line A B changing its position, relatively to that plane. This line A B is therefore perpendicular to all the lines drawn in the plane X, from its foot, since the sides A E and

Fig. 125.

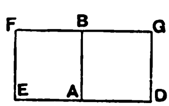
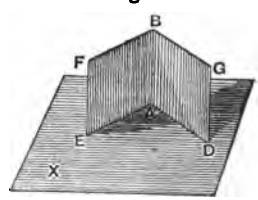


Fig. 126.

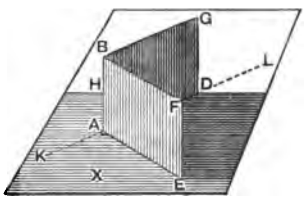


A D of the folded rectangle can be applied successively to each of these lines, by the movement just described.

8.

From the preceding construction is derived a convenient practical method of raising from a given point upon a plane a line perpendicular to that plane, or for letting fall from a point outside of a plane a line perpendicular to that plane. If the given point is in the plane, at A, for example, or outside of the plane as at H, the rectangle E F B G D A can be moved along the plane X, till the fold A B touches the given point; and A B will become in either case the perpendicular required.

Fig. 127.



9.

It follows also that the line A B is perpendicular to the plane X when it is perpendicular to two lines A E and A D on that plane; for then A B may be regarded as the fold of the rectangle of which one of the folded sides is coincident with A E, and the other with A D, and the fold could not fail to be perpendicular to the plane X.

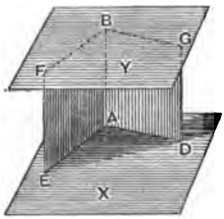
10.

If it were desired to raise upon any line K L a plane perpendicular to the plane X in which that line is, the folded rectangle G B F E A D can be still used, for it is only necessary to put upon the line K L the side A D of one of the parts A D G B of the folded rectangle, and the plane of that part A D G B will be that which is required.

11.

It is readily seen that if a third plane Y be placed upon the two sides E B and B G of the folded rectangle, that plane Y would also be perpendicular to the line A B, and consequently parallel to the plane X.

Fig. 128.



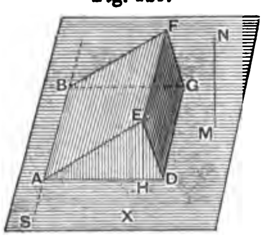
Therefore, if to a plane X be raised three perpendiculars E F, A B, and D G, of equal length, which are not on the same straight line, the plane Y, which passes through the three points F, B, and G, will be necessarily parallel to the plane X.

12.

When two planes are not parallels, it is easy to ascertain the angle which they make with one another by means of the folded rectangle.

To do this, we apply one of two parts A B G D of that rectangle upon the plane X, and it is evident that E A D, or its equal F B G, will measure the inclination of the plane E A B F to the plane D A B G. Observing that A B is the common section of these planes, and that E A and A D are each perpendicular to A B, we deduce the following rule: Two planes which are not parallel being given, it is necessary first to find the straight line which is their common section; afterwards, from any point in that line, to draw two perpendiculars each in one of the two planes, and the angle contained between these two perpendiculars measures the angle which the two given planes make with each other.

Fig. 129.



13.

During the movement of A B F E round the fold A B, the line A E the end E of which describes a circular arc, E D obviously remains always in a plane E A H D perpendicular to the plane X, and the inclination of the line E A to the plane X is precisely the angle E A D. Hence it is readily seen that the inclination of any straight line E A to the plane X is measured by the angle E A H, contained between that line and the line A D, which passes through A and through that point H of the plane X, which is the intersection of a perpendicular E H, let fall upon the plane from any point E on the line A E.

14.

The mere inspection of the figure employed in the preceding paragraphs furnishes a new method of letting fall from a point E, outside the plane X, a line E H, perpendicular to that plane.

Having drawn any line B A S in the plane X, we draw from the given point E, a perpendicular E A to that line. Then from the point A, where that perpendicular falls, we draw in the plane X the line A D perpendicular to A B, and from the given point E drawing E H perpendicular to A D, that line is perpendicular to the plane X.

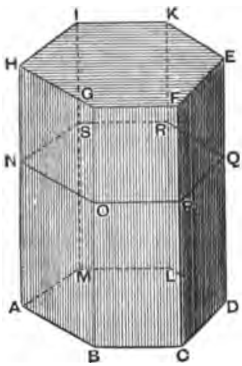
15.

Hence also is deduced a second way of raising M N perpendicular to a plane X, from a point M on that plane.

Having drawn a perpendicular E H, from any point E, outside the plane X, we draw through the given point M a straight line M N, parallel to H E, and it will also be perpendicular to the plane X.

16.

Fig. 130.



After the parallelopiped, the most simple solid is the right prism. It is a figure A B C D E F G H I K L M having for its two opposite bases two equal polygons, so placed that the sides G F, F E, &c., of the one are parallel to the sides B C, C D, &c., of the other, the other faces being rectangles A B G H, B G F C, &c.

17.

Geometers suppose these figures formed like the parallelopiped, by a base A B C D L M moving parallel to itself, with its angles A, B, &c., following the lines perpendicular to the plane of the base.

18.

The different kinds of right prisms are distinguished by the names of the figures which constitute their bases. The hexagonal prism, for instance, is that having for its base a hexagon.

19.

In finding means of measuring all kinds of right prisms, it is to be observed first that of two right prisms having equal bases, that which has the greater height is greater in solidity in proportion as its height is greater.

20.

It is farther to be observed that two right prisms which have the same height, but of which the one has a base containing a certain number of times the base of the other, would be in the same proportion as their bases. The truth of this proposition is readily perceived on considering the formation of the prisms explained in paragraph 17. Let *a b c d e f g h i k l m* and A B C D E F G H I K L M be two prisms which have the same height, and let the base *a b c d l m* of the less be, for instance, one quarter of the base of the greater A B C D L M. Seeing that the two prisms

Fig. 131.

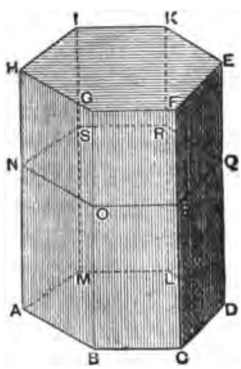
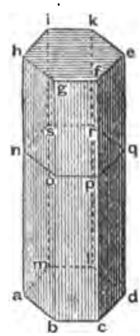


Fig. 132.



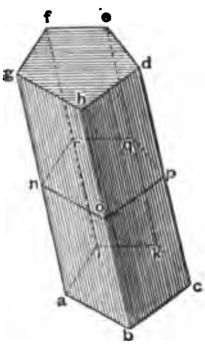
are produced by the movements of these two bases, it follows that any plane whatever, parallel to the planes of the two bases, will cut in the two prisms two polygons of which each will be equal to the base of the prism in which it is cut, that is to say, that the section of the large prism will be always four times that of the small prism: therefore the prism A B C D E F G H I K L M may be regarded as composed of slices all quadruples of those of the prism *a b c d e f g h i k l m*; and consequently the solidity of the first prism will be quadruple that of the second.

21.

After these two remarks it is not difficult to form the following rule for measuring all right prisms.

Measure first in square feet, or in square inches, &c., the area of the base of the proposed prism, then multiply this area by the number of feet or inches, &c., contained in the height of the prism, and the product will give the number of cubic feet or cubic inches &c., contained in the proposed prism, will be, consequently, its measure.

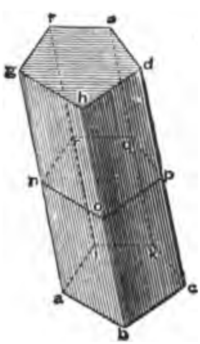
Fig. 133.



22.

The name *prism* is applied to solids which have two bases equal polygons like the preceding, but of which the other faces are parallelograms instead of being rectangles. Such prisms are distinguished from those already described by the name *oblique prisms*, in opposition to those named *right prisms*.

Fig. 134.



23.

An oblique prism is conceived to be formed by a base *a b c k i* which moves parallel to itself, its angles following parallel lines *ag*, *bh*, *cd*, &c.; rising from the plane of the base, but not perpendicularly to it.

24.

The analogy existing between this formation and that of right prisms (mentioned in Par. 16) suggests means of measuring the solidity of oblique prisms; for if we imagine at the side of an oblique prism *a b c d e f g h i k* a right prism A B C D E F G H I K having the same base, and both prisms included between two parallel planes, we see that the solidity of these two bodies is absolutely the same.

For, if through any point P of the height a plane is drawn parallel to the base, the sections N O P Q R and *n o p q r*, formed by

Fig. 135.

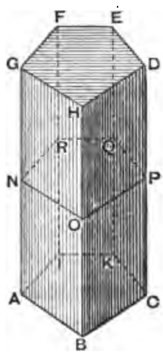
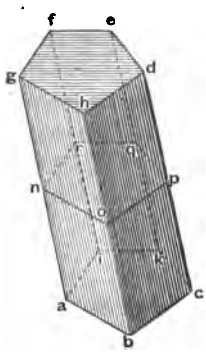


Fig. 136.



this plane in each of the two prisms, may be regarded as the equal bases A B C K I and *a b c k i* arrived at N O P Q R and *n o p q r* in the course of the movement which forms these two prisms, and these two sections are therefore equal polygons. Now if all imaginable slices that can be formed in these two prisms by like cutting planes are equal, the total of these slices, that is to say, the whole prisms, must necessarily be equal.

This proposition is usually thus stated:—Oblique prisms are equal to right prisms having the same base and the same height. The height of a prism is the perpendicular let fall from the uppermost plane to the lowest plane, or upon its extension.

25.

As parallelopipeds are in the class of prisms, what has been said as to prisms applies also to oblique parallelopipeds, that is to say, to figures *a b c d e f g h* produced by moving a square, a rectangle, or even a parallelogram in such a manner that its four angles follow parallel lines rising obliquely from the base.

Accordingly, the oblique parallelopiped *a b c d e f g h* is equal

Fig. 137.

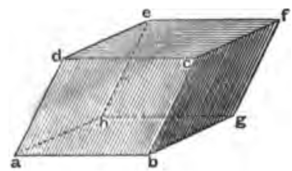
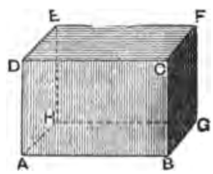


Fig. 138.

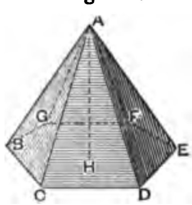


to the right parallelopiped A B C D E F G H if the base *a b g h* is the same, or has the same area, as the base A B G H, and if the perpendicular let fall from the plane *d c f e* upon the plane *a b g h* is equal to the perpendicular let fall from the plane D C F E upon the plane A B G H.

26.

Having discussed parallelopipeds and prisms, we will now examine pyramids, that is to say, such bodies as A B C D E F G, enclosed by a certain number of triangles which have all a common vertex A, and which terminate at any base B C D E F G. It is necessary to consider solids of this kind, not only because they are met with in buildings and in other structures, but because all solids bounded by planes are assemblages of pyramids, as rectilineal figures are assemblages of triangles.

Fig. 139.



To be assured of this, we need only draw from a point taken where we please, in the interior of the proposed body, lines to all the angles of the body.

27.

Pyramids like prisms are distinguished by the names of the figures which form their bases.

28.

When the base of the pyramid is a regular figure, and its apex is perpendicularly above the centre H of its base, as in figure 139, the pyramid is then called a *right pyramid*; it is named on the other hand an *oblique pyramid*, when the apex is not perpendicularly above the centre of the base, as in figure 140.

29.

In ascertaining a method of measuring all kinds of pyramids, whether right or oblique, we may begin by making a few general observations on these figures, suggested by a knowledge of the properties of prisms.

Noticing the equality of prisms which have the same base and the same height, we naturally recall the fact that parallelograms are also equal when they have these same conditions, and it is also the same with triangles. When these three truths are at once presented to the mind, analogy would indicate that the properties which are common to parallelograms and triangles, may be also common to prisms and pyramids, we may therefore conjecture that pyramids which have the same base and the same height have the same solidity.

30.

The following reflections will confirm this conjecture.

Fig. 140.

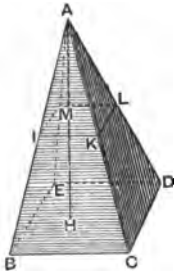
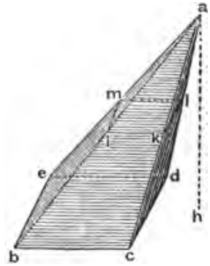


Fig. 141.



Let ABCDE and *abcde* be two pyramids of which the heights A H and *a h* are the same, and of which the bases are two equal figures, for instance two equal squares B C D E and *b c d e*: if these two pyramids are considered to be cut by an infinite number of planes parallel to their bases, it is readily seen that these slices of pyramids will give the equal squares I K L M, and *i k l m*, and consequently the two pyramids may be regarded as assemblages of a like number of slices of which

Fig. 142.

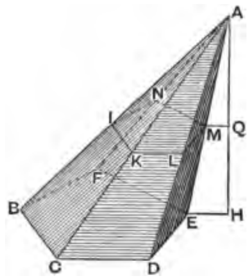
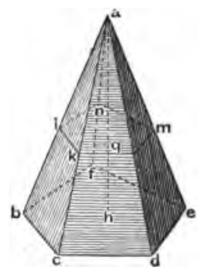


Fig. 143.



those that correspond in the two pyramids are equal. It will therefore be concluded that the total of the slices is in each case the same, that is to say, that two pyramids have the same solidity.

If the bases of the two pyramids were other regular polygons or irregular polygons B C D E F and *b c d e f*, equal to each other, all the slices I K L M N and *i k l m n* of these two pyramids must still be considered equal respectively, and hence it must be concluded that pyramids have always the same solidity when they have equal bases and heights.

31.

All this is easily conceived after the demonstration that has been given of the equality of prisms which have the same height; yet the similarity of any section I K L M N of a pyramid to the base B C D B E F, and the equality of the sections I K L M N and *i k l m n* are among propositions which, though generally obvious, may certainly demand demonstration.

To find this demonstration we must give some consideration to questions as to the similarity of solid figures.

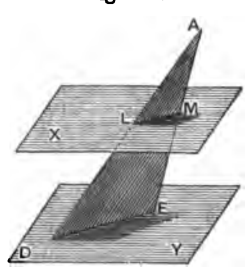
32.

Supposing the pyramid A B C D E F cut by a plane I K L M N parallel to the base, we have to prove that the section of the pyramid formed by this plane is a polygon perfectly similar to the polygon B C D E F, and that the pyramid A I K L M N is itself similar to the pyramid A B C D E F, that is to say, that the angles formed by all the lines of these two figures are respectively equal, and that all the sides of the small pyramid have the same proportions as those of the larger.

33.

We observe in the first place that if two planes X and Y are parallels, and if any two lines A L D and A M E, extending from the same point A, cut these two planes, the straight lines L M and D E joining the points of intersection L, M, D, and E, are parallel. For if these two lines were not parallel, they would meet somewhere, when prolonged; but if they should so meet the planes in which they are situated, and out of which they cannot pass, however much prolonged, would also meet and could not therefore be parallel as supposed.

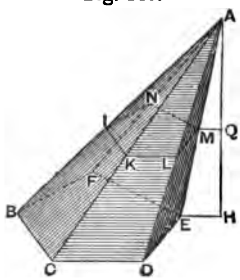
Fig. 144.



34.

If then the plane I K L M N be supposed parallel to the plane B C D E F, it follows that all the lines M L, L K, K I, I N, and N M, are parallel to the lines E D, D C, C B, B F, and F E, and consequently that the triangles, A L M, A K L, A I K, &c., are similar to the triangles A D E, A C D, A B C, &c. If we take one of the sides of these triangles, A M, for instance, as a scale, or common measure of all the sides of the smaller pyramid, whilst the corresponding side A E serves as a scale for the sides of the larger pyramid, it is at once seen that the sides M L, L K, K I, &c., of the polygon I K L M N, are proportional to the sides E D, D C, C B, &c., of the polygon B C D E F.

Fig. 145.



It is also obvious that all the angles I K L, K L M, &c., must be respectively equal to the angles, B C D, and C D E, since the former are contained by lines respectively parallel to those containing the latter. The two polygons I K L M N and B C D E F are therefore similar.

35.

Again as the sides A M, A L, A K, &c., are proportional to the sides A E, A D, A C, &c., and the angles A L M, A L K, &c., are respectively equal to the angles A D E, A D C, &c., because of the similarity of the triangles A L M, A D E, A L K, A D C, &c., the two pyramids A I K L M N, and A B C D E F, are similar in all respects.

36.

Finally, if from the point A a perpendicular A H be drawn to the plane of the polygon B C D E F, and if Q be the point where this perpendicular cuts the plane of the polygon I K L M N, it is obvious that the lines A Q, and A H, the heights of the two pyramids A I K L M N, and A B C D E F, respectively, have the same ratio to one another as the corresponding sides, A M, A E, A L, A D, &c., or what comes to the same, that if the heights A Q, and A H, be taken for the scales of two pyramids, the sides A M, A L, &c., contain as many parts, of A Q, as the sides A E, A D, &c., contain parts of A H.

37.

Considering now the two pyramids A B C D E F and *a b c d e f* together, we see that the two sections I K L M N, *i k l m n*, being respectively similar to the bases B C D E F and *b c d e f*, which are like, will also be like. These two sections must also be equal, since the scales of these two figures are the equal lines A Q, and

Fig. 146.

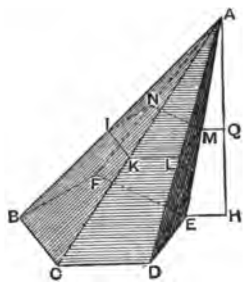
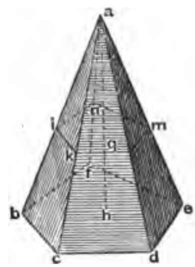


Fig. 147.



a q, the heights of the pyramids A I K L M N, and *a i k l m n* respectively.

Thus, without knowing what is the actual solidity of pyramids, we know certainly that if they have the same height and the same base they are equal, as we had conjectured (Par. 29).

38.

If the bases of two pyramids, instead of being the same, were merely equal in area, the pyramids would still be equal in

Fig. 148.

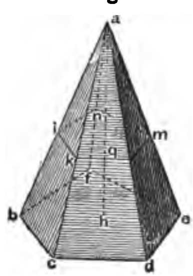
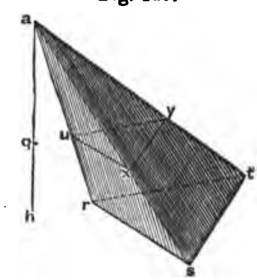


Fig. 149.



solidity. For let *a b c d e f* and *a r s t* be two pyramids which have the same height *a h*; if these two pyramids were cut by any plane parallel to the base, it is evident that there is the same relation between the area *i k l m n*, and the area *b c d e f* as between the area *u x y* and the area *r s t*, since *i k l m n* and *b c d e f* being (Par. 34) similar figures, they differ only (1st Part, Par. 48) in their scales *a q*, *a h*, &c., and the figures *u x y* and *r s t* being also similar, they differ only in their scales, which are still the lines *a q* and *a h*.

But if the bases *r s t* and *b c d e f* are equal in area, their proportional parts *u x y* and *i k l m n* must be equal: therefore all the slices of two pyramids *a r s t* and *a b c d e f* must have the same extent, and consequently their totals, that is to say, the pyramids themselves, are equal in solidity.

39.

If the base *b c d e f* of the first pyramid contained the base *r s t* a certain number of times, the solidity of the first pyramid *a b c d e f* must contain the same number of times the solidity of the second *a r s t*.

For in this case, the base *b c d e f* being divided into several parts, each equal to the base *r s t*, the pyramid *a b c d e f* might be conceived to be composed of several other pyramids having for their bases the parts of *b c d e f*. Now each of these new pyramids would be equal to the second pyramid *a r s t*, as proved in the preceding Paragraph: therefore, &c. If the base *r s t* were not exactly contained in the base *b c d e f*, but if these two bases had a common measure x, each of the two bases *b c d e f* and *r s t* might be divided into parts each equal to x, and then the two pyramids *a b c d e f* and *a r s t* would be composed of as many new pyramids, all equal to each other, as the number of parts x, contained in their respective bases.

Therefore the pyramids *a b c d e f* and *a r s t* are to one another as their bases.

If the bases were incommensurable, it would be seen, notwithstanding, that the pyramids would be in the ratio of their bases, by reasoning similar to that employed in the like case

(2nd Part, Par. 28) of comparing figures having their sides incommensurable; that is to say, the measure X might be infinitely diminished, so that it might be treated as a common measure of the base *r s t* as well as of the base *b c d e f*.

40.

Having ascertained that pyramids which have the same height, have the same ratio as their bases, we should feel that the measurement of their solidity should present little difficulty. For we have only to learn how to measure a single pyramid in order to measure all others. Let us suppose, for example, that we knew how to measure the pyramid A B C D E,

Fig. 150.

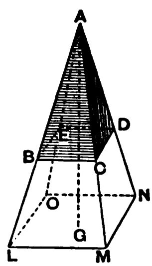
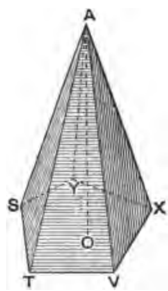


Fig. 151.



and that we were required to measure the pyramid A S T V X Y, which has neither the same height nor the same base as the former: we should begin by making a pyramid similar to the pyramid A B C D E, but having the height of the pyramid A S T V X Y, which could be easily done, for it would suffice (Par. 35) to prolong the sides A B, A C, A D, and A E, and to cut them by the plane L M N O, at a distance A G from the apex A equal to the height A O.

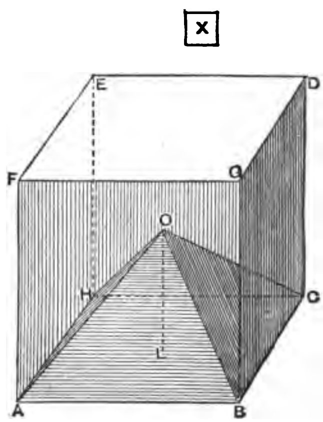
This done, since by the supposition, we can measure the pyramid A B C D E, it is obvious that we can also measure the pyramid A L M N O, which is similar to it, for whatever be the operations in measuring A B C D E, we can employ the same operations in measuring the similar pyramid A L M N O, taking care to employ a different scale for the latter.

Supposing then that the pyramid A L M N O is measured, its measure will determine also that of the proposed pyramid A S T V X Y, for, by the preceding paragraph, these two pyramids are to each other as their bases L M N O and S T V X Y, and we have been taught in the second part to find the ratio of these two bases.

41.

Since then it is only necessary to know how to measure a single pyramid in order to know how to measure all other imaginable pyramids, let us take a very simple case, forming a pyramid by drawing from the four angles A, B, C, and H, of one

Fig. 152.



of the faces of a cube A B C D E F G H, four lines to the point O, the centre of that cube, that is to say, the point equally distant from A, D, B, E, etc. It is readily seen that this pyramid is the sixth part of the cube, since the cube could be cut into six similar pyramids, taking each face for a base. Now, the solidity of the cube is the product of the height A F by the base A B C H: therefore, to have the solidity of the pyramid, we divide the product of A F by A B C H, into six equal parts, or, what comes to the same, we multiply the sixth part of the height A F by the base A B C H, and as the sixth part of the height A F is the third part of the height O L of the pyramid O A B C H, seeing that its height O L is half the side of the cube, it follows that the measure of the pyramid O A B C H is the product of one third of its height by its base.

42.

Let us suppose now that we have to measure any pyramid whatever, O K M N S T V. We can conceive a cube of which the side A B or A F is double the height O L of the proposed pyramid,

Fig. 153.

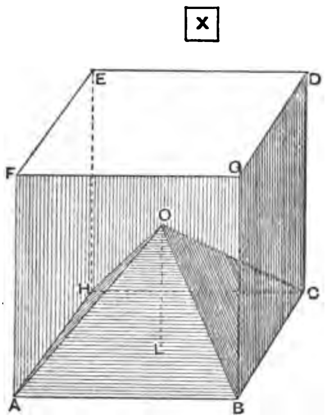
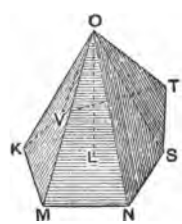


Fig. 154.



and we can conceive in this cube a pyramid O A B C H having its apex at the centre, and having for its base one of the faces A B C H of the cube. This new pyramid will have the same height as the former, and consequently (Par. 39) the solidity of O A B C H will be to that of O K M N S T V as the base A B C H to the base K M N S T V. Now, by the preceding paragraph, the product of one third of the common height O L by the base A B C H is the solidity of the pyramid O A B C H; therefore, the product of one third of the same common height O L by the base K M N S T V will be the solidity of the proposed pyramid O K M N S T V.

And thus is ascertained the general theorem that a pyramid has for its measure the product of its base by one third of its height.

43.

As we have seen (Par. 21) that the solidity of a prism is the product of its base by its height, it is clear, by the preceding paragraph, that a pyramid is always one third of a prism having same base and the same height.

44.

Having measured all solids bounded by planes, we proceed to find one way for the measurement of solids having curved surfaces. And as we have treated in the third Part only of figures having circular outlines, we shall now examine only solid bodies of which the curvatures are circular.

In the examination of these bodies we shall have two objects, the measurement of their surfaces and that of their solidities; for these surfaces being, either wholly curved, or partly plane and partly curved, we cannot refer their measurement to the first Part, as we have done with bodies bounded by planes.

45.

The simplest of all curved solids is the cylinder. It is a body

Fig. 155.

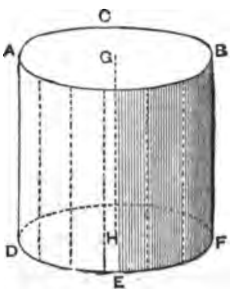
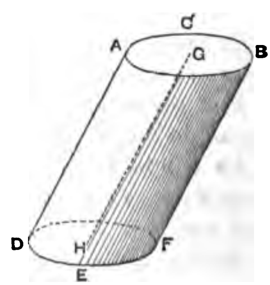


Fig. 156.



like A B C D E F, of which the two bases A B C and D E F are two circles, equal and parallel, joined by a curved surface which may be conceived to be a plane folded round their circumferences.

When the two circles are so placed that the centre G of the first is perpendicularly above the centre H, of the second, the cylinder is named a *right cylinder*.

The cylinder is on the other hand called *oblique* when the line drawn through the two centres G and H is inclined to the planes A B C and D E F.

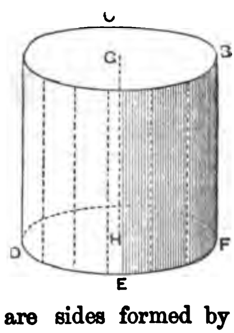
46.

The geometrical formation of these solids, analogous to that of prisms and parallelopipeds (noticed in Par. 17), consists in moving a circle parallel to itself, in such a way that all its points describe parallel straight lines rising above the plane of that circle.

47.

We arrive, in the following manner, at the measurement of the surface of a right cylinder, which is often necessary in practice.

Fig. 157.



The two circumferences A B C and D E F being divided each into the same number of equal parts, the points of division of the one being perpendicularly above those of the other, let us draw straight lines to join the corresponding angles of the two regular polygons presented by this operation. It is obvious that we then have a prism the surface of which will consist of as many rectangles as there are sides formed by the division of each of the circumferences A B C and D E F. Now, all these rectangles having the same height A D, their total measure will be the product of the height A D, by the sum of all the bases, that is to say, by the outline of the enclosed polygon inscribed in the circle D E F or A B C.

But since in proportion as the number of the sides of this polygon is increased, the outline of the polygon approaches more nearly to the circumference of the circle, and the surface of the prism approaches to coincidence with that of the cylinder, it follows that if the number of sides of the polygon be conceived infinite, the prism will in no way differ from the cylinder. The curved surface of a right cylinder is therefore equal to a rectangle of which the height is A D, and the base a line equal to circumference D E F.

This proposition enables us to find, for instance, how much stuff would be required for covering a cylindrical pillar or for lining the interior of a *round* tower.

48.

As to the surface of an oblique cylinder, it cannot be measured in the same way, because, instead of rectangles, we should have parallelograms of different heights. It is only by methods very complex and difficult that even the approximate value of such surfaces has been ascertained, and problems of this kind are not within the scope of these Elements.

49.

The solidity of cylinders, whether *right* or *oblique*, can be readily found; for it is evident that all that has been said as to prisms applies equally to cylinders, if cylinders are conceived to be the ultimate prisms that can be inscribed in them.

Accordingly, cylinders having the same base and the same height are equal in solidity.

50.

And the solidity of any cylinder is the product of its base by its height.

51.

The cone is the most simple curved solid after the cylinder; it is a figure like A B C D E, of which the base is a circle and of which the surface is composed of an infinitude of straight lines, drawn from the circumference of the base and

Fig. 158.

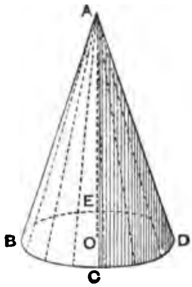
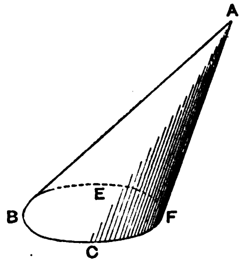


Fig. 159.



meeting at the apex. This solid may be regarded as a pyramid having a circle for its base.

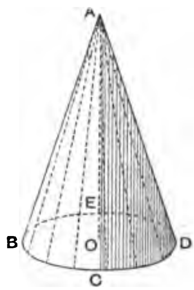
52.

If, as in figure 158, the point or apex A of the cone is perpendicularly above the centre O of the base, the cone is named a *right cone*; and it is called *oblique*, if the apex be not perpendicularly above the centre of the base, as in figure 159.

53.

In order to measure the surface of a right cone A B C D E, we may regard it as the ultimate pyramid that can be inscribed in it, that is to say, we may divide the circumference of its base B C D E as we did with the circumference of the cylinder, into an infinity of small sides, and, drawing lines from all the angles of this polygon to the apex A, we should find the conical surface to be an assemblage of small isosceles triangles, of which the height is equal to the side A B of the cone, and the sum of the bases of which is equal to the circumference B C D E; hence it is readily seen that the measure

Fig. 160.



of that surface is found by multiplying the half of A B by the circumference B C D E.

54.

Remembering now that the area of a sector of circle is (3rd Part, Par. 10) the product of the arc of the sector by half the radius, we see that, in order to cover the right cone A B C D E

Fig. 161.

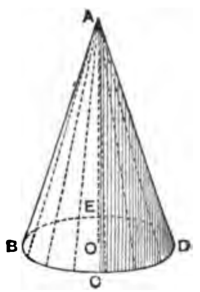
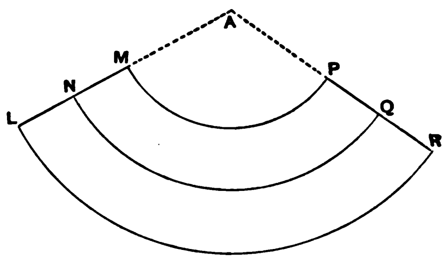


Fig. 162.



with a flexible surface, such as paper, &c., it would be necessary to take a sector of a circle A L R of which the radius is equal to A B, and of which the arc is equal to the circumference B C D E.

55.

When the cone is oblique, it is very difficult to ascertain the measure of its surface, like that of the oblique cylinder, even approximately, and this problem is beyond the scope of these Elements.

56.

With respect to the solidity of a cone, either *right* or *oblique*, it may be regarded as the ultimate pyramid that can be inscribed in it, and it may consequently be treated generally as a pyramid. Accordingly, cones that have the same base and same height are equal.

57.

And the solidity of any cone is the product of its base by one third of its height.

58.

Sometimes it is required to measure a body like B C D E F G H, which is called a truncated cone, or the *frustum of a cone* : it is the part which remains of a cone A F G H, when a smaller cone

Fig. 163.

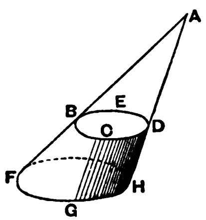
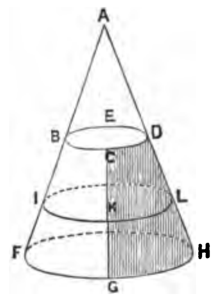


Fig. 164.



A B C D E has been cut from it, by a section parallel to the base F G H. It is obvious that the measure of such a solid is the difference between the solidities of the two cones, A B C D E, and A F G H.

59.

The surface of a truncated cone, formed by the cutting of a right cone, may be found more simply than by measuring separately the surfaces of the two cones, and deducting the one from the other. The following method may be employed, which can be readily understood after what has been said (Par. 54).

Let A L R be the sector that it would be necessary to construct in order to cover the cone A F G H. Describing from the centre A, and with radius A M equal to A B an arc M P, the space M P R L is evidently a portion of a ring suited to cover the surface of the truncated cone. If, now, the two circumferences of which M P and L R are similar arcs be completed, there is a complete ring of which the measure (3rd Part, Par. 8) is the product of M L, equal to B F, by the circumference of which A N is the radius, N being the middle of M L. Therefore the portion

Fig. 165.

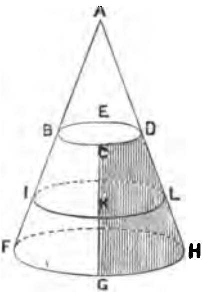
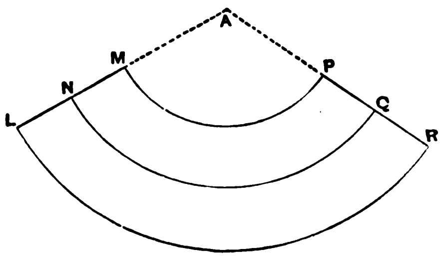


Fig. 166.



of ring M P R L, or the surface of the truncated cone B C D E F G H, which is its equal, is measured by multiplying M L by the arc N Q, or, what comes to the same thing, by multiplying B F by the circumference I K L of the section of the solid proposed, formed by a plane parallel to the base, passing through the middle I of the side B F.

60.

The last of the solid bodies of which we shall treat is named a *sphere* or *globe*: it is that the surface of which is at every point equally distant from one point which is in the centre. The surface of this figure has often to be measured. It may be required to ascertain, for example, how much gilding is necessary for a ball, how many sheets of lead are required for the covering of a dome, &c.

61.

Let X be the sphere the surface of which is to be measured. This solid can obviously be conceived to be produced by the revolution of a semi-circle A M B round its diameter A B.

Supposing first that, instead of the semi-circle we had a regular polygon of an infinite number of small sides, or, if preferred, of a very great number of sides, let us propose to measure only the surface Z produced by the revolution of that polygon.

Fig. 167.

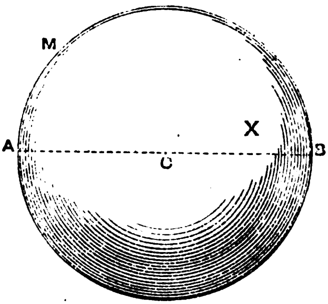
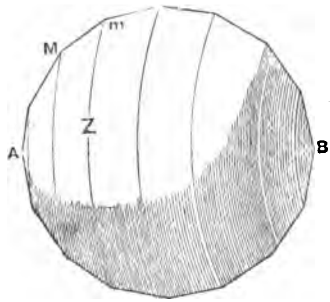


Fig. 168.

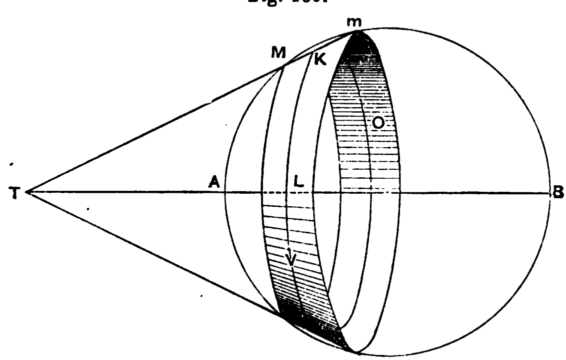


It will then be easy to pass from the measurement of that surface to the measurement of the surface of the sphere, as we passed from the measurement of rectilineal figures to that of the circle.

62.

To measure the surface of the solid Z let us examine the

Fig. 169.



small part of that surface produced by a single side M m of the inscribed polygon while it turns round the diameter A B. It is obvious that this side M m describes in that movement the surface of a truncated cone V; for prolonging the line m M till it meets in T the diameter or axis of revolution A B, if that line T M m were to turn in the same time as the semi-circle A M B, it would obviously describe a right cone, having the apex T, and for its base the circle described by the point m; the surface V, produced by the movement of M m, is a slice of that cone, included between the planes of the circles which the points M and m describe in their revolution. But as we have seen (Par. 59), the surface V is equal to a rectangle of which M m is the height, and the base a line equal to the circumference K L O, described by the point K, the middle of M m; therefore the surface produced by the revolution of the polygon is equal to the sum of as many rectangles of that nature as there are sides in the polygon, such as M m.

Now, as all the sides M m, the heights of these rectangles, are supposed equal, we may regard the surface sought as a whole rectangle having the height M m, and a base equal to the sum of all the circumferences such as K L O, that is to say, such as are described by the middle point of each of the small sides.

But the polygon inscribed in the semi-circle A M B, having a very great number of sides, the smallness of the height M m, and the excessive greatness of the base, render it impossible to construct such a rectangle.

To obviate this difficulty, it is easy to conceive all the small rectangles changed into others having all the same height, not imperceptible, like M m, but of such magnitude that each of the bases may be very small: in this way, the addition of all the small bases will only make a length comparable to the height.

63.

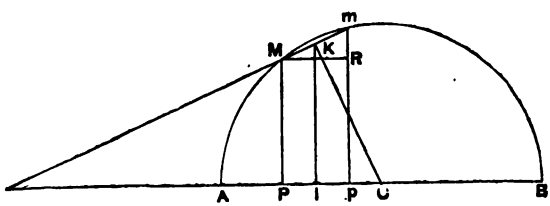
Let us see, then, if we cannot thus change our small rectangles.

Suppose first, to simplify the problem, that our rectangles, instead of having for their bases lines equal to the circumferences K L, have for bases only the radii K I of these circumferences.

It will not be difficult afterwards to apply what we find for these rectangles to the rectangles in question.

We have then to find a rectangle having for its measure the product of M *m* by K I, and for its height some lines incomparably greater M *m*, always the same wherever the small side M *m* may be situated. Let us choose, for example, the line C K, the apotheme of the polygon of which M *m* is the side, this line being always the same for every side of the polygon. We have then to seek for a line such that its product by C K is equal to the product of K I by M *m*, that is to say (2nd Part, Par. 7), we have to find a fourth proportional to the three lines C K, K I, and M *m*. We know that it is by means of similar

Fig. 170.



triangles that lines are found proportional to other lines: we must therefore form similar triangles, having for their corresponding sides the lines in question; this is done by drawing M R perpendicular to *mp*. We have then the triangle M *m* R and K I C which are similar, for they have each a right angle, the one at R, the other at I, and further they have the equal angles *m* M R and I K C, because the former makes a right angle when added to the angle M *m* R, which is equal to the angle M K I, and the other I K C also makes a right angle with M K I.

Hence it is readily concluded that C K is to K I as M *m* to M R; that is to say that M R is the fourth proportional required, or, what comes to the same thing, that the rectangle of C K by M R or by P *p*, is equal to the rectangle of M *m* by K I.

But as the rectangle that we first proposed to change was not that of M *m* by K I, but was that of M *m* by the circumference of which K I is the radius, we may remember that, as circumferences are as their radii, the equality of the rectangle of M m by K I to that of P p by C K implies the equality of the rectangle of M m by the circumference of K I to the rectangle of P p by the circumference of C K. For it is obvious that if two rectangles are equal, and that, while their heights remain the same, their bases are proportionally increased, the rectangles so increased are also equal.

64.

Having discovered, in the two preceding paragraphs, that all the small truncated conical surfaces, such as V (Fig. 169), are equal to as many rectangles having each for its height a line equal to the circumference of which C K is the radius, and having each for its base a small line P p corresponding to each of the sides M m, we infer that the sum of any number of these small surfaces, for example those extending from A down to p, is equal to a rectangle having for its height a line equal to the circumference of C K, and for its base the sum of all the lines such as P p, extending from A down to p, that is to say, the line A p.

Therefore to have the whole surface produced by the revolution of the entire polygon, it is necessary to make a rectangle having for its base the circumference described with the radius C K, and having for its height the diameter A B.

65.

It is now easy to measure the surface of the sphere: for it is clear that the more sides the polygon has, the more nearly does the solid produced by its revolution approach to equality with the sphere, and the more nearly also does the apotheme C K approach to equality with the radius; so that if the polygon is conceived to become a circle, the apotheme C K becomes the radius itself, and the surface of the sphere has the same area as a rectangle having for its height the diameter, and for its base a line equal to the circumference of the circle which produces it, and which is usually called a great circle of the sphere.

66.

The curved surface of a spherical segment A M L N O, that is to say of the part cut off from a sphere by a plane M L N O perpendicular to the diameter, is measured by the product of its depth or verted sine A P by the circumference of the

Fig. 171.

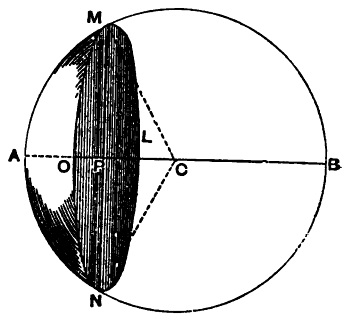
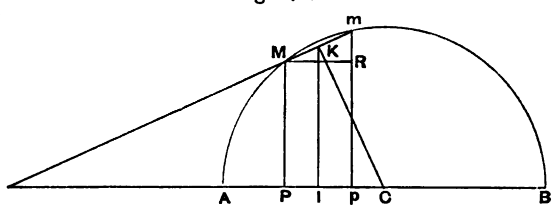


Fig. 172.

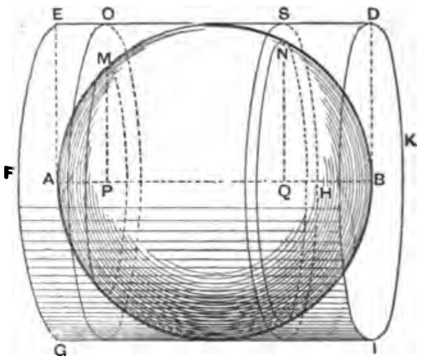


great circle A M B N. The reason for this is the same as that proving (Par. 64) that the sum of the surfaces of all the small truncated cones, included between A and *m*, is equal to the rectangle of height A *p* and base the circumference of which C K is the radius.

67.

The preceding measurement of the surface of a sphere shows that if the rectangle A B D E were at the same time as the semicircle A M N B caused to revolve round A B, the curved surface of the right cylinder E F G I K D H, produced by the revolu-

Fig. 173.



tion of that rectangle, is equal to that of the sphere described by the semicircle, which is usually thus expressed :—The surface of a sphere is equal to that of the circumscribed cylinder.

68.

If the cylinder and the sphere are at the same time cut by two planes perpendicular to the diameters A B at P and Q, the segments of the cylindrical and of the spherical surfaces produced by the revolution of the straight line O S and of the arc M N are equal in area.

69.

It is also manifest, from the preceding, that the surface of a sphere is equal to four times the area of its great circle ; for the surface of this great circle has for its measure the product of half the radius, or a quarter of the diameter by the circumference, and the surface of the sphere is equal to the product of the whole diameter by the same circumference.

70.

The measurement of the surface of the sphere being found, it is easy to measure its solidity; for a sphere may be considered to be an assemblage of an infinite number of small pyramids, having their summits at its centre and having their united bases extending over the whole surface. Now, as each of these pyramids has for its measure the product of one third of its height, that is, of the radius by its base, their sum total or the solidity of the sphere is measured by multiplying one third of the radius by the surface of the sphere, that is to say, by four times the area of the great circle.

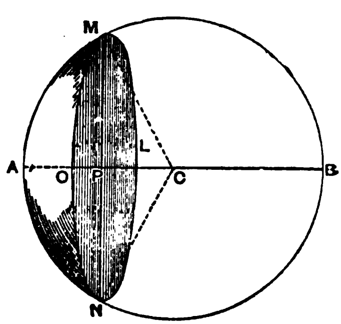
71.

As the product of one third of the radius by four times the great circle is the same as the product of four times one third of the radius, that is to say, of two thirds of the diameter, by the great circle, and as the solidity of the cylinder E F G I K D H has for its measure the product of the diameter by the same great circle which is its base, it follows that the solidity of the sphere is two thirds of that of the circumscribed cylinder.

72.

If the solidity of a spherical segment A M L N O were to be measured, it would evidently be necessary, in the first place to measure the portion of the sphere produced by the revolution of the sector C A M; which is done by multiplying one third of the radius by the surface of the segment A M L N O, then to subtract from that measure the solidity of the cone produced by the revolution of the triangle C P M, that is to say, the cone having for its base the circle M L N O, and

Fig. 174.



C P for its height, and the remainder would be the required solidity of the segment.

73.

We will conclude these Elements by several propositions as to the solidity and the surface of similar bodies. These propositions naturally present themselves when we reflect upon what constitutes the similarity of two bodies. It may even be said that they could scarcely fail to be discovered by analogy, on recalling what has been said (1st Part, Par. 34 and following) as to the similarity of plane figures, that is to say, of those described upon planes.

We have determined (Par. 32) in what consists the similarity of two pyramids: the definition there given of similar pyramids can be extended to all bodies bounded by planes, that is to say that two bodies of that kind are called *similar*, when all the angles formed by the sides of the first are the same respectively as the angles formed by the sides of the second, and when the sides of the one body are proportional respectively to the homologous sides of the other.

74.

As to bodies which are not bounded on all sides by planes, cylinders and cones, for example, it is also easy to determine the conditions necessary to render them similar.

Two right cylinders are similar when their heights have the same ratio as the radii of their bases.

75.

If the cylinders are oblique, it is necessary besides that the lines which join the centres of the two circles, in each of these cylinders, make the same angles with the planes of their bases.

76.

The same definitions can be applied to cones, by putting, instead of the line which passes through the centres of the two bases of the cylinder, that which goes from the apex of the cone to the centre of its circular base.

77.

In order that two truncated cones may be similar, it is necessary, in the first place, that the cones of which they are portions should be similar; and in the second place, that their heights should be to each other as the radii of their bases.

78.

With respect to spheres, it is clearly seen that they are all similar to one another, like all figures, either solid or plane, which require only one line to determine them, as the circle, the square, the equilateral triangle, the cube, the cylinder circumscribed about the sphere, &c.

79.

Generally, it may be said of similar solids, as it has been said of similar plane figures, that they differ only in the scale on which they are constructed. This explanation, if well considered, leads to two fundamental propositions as to the surface and the solidity of similar bodies.

80.

The first proposition teaches that the surfaces of two similar solids are as the squares of their corresponding sides; that there is, for example, the same relation between the surfaces of two similar pyramids *z*, and Z, as between the squares *a b c d*, and A B C D, made upon the sides *a b*, and A B, which correspond in these two pyramids.

To discover this proposition, there is only required the reasoning employed (1st Part, Pars. 43 and 44), that is to say, if P be considered the scale of the pyramid Z, and *p*, the scale of the similar pyramid *z*, the lines used for measuring the surface of Z, and that of the square A B C D, contain P as many times as there are parts *p* in those used for measuring the surface of *z*, and of the square *a b c d*.

It follows from this that the product of the lines entering into the measurement of Z, and of A B C D, give the same number of squares X, made upon P, than the product of the lines used to measure *z*, and *a b c d*, give of squares *x* made upon *p*, that is to say that the numbers expressing the relation of the surface of

Fig. 175.

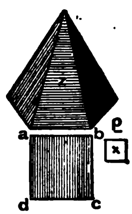
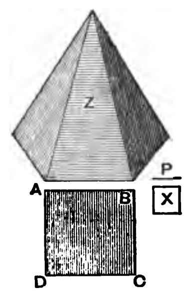


Fig. 176.



the pyramid Z to the square A B C D, are the same as those which express the relation of the surface *z* to the square *a b c d*.

The same reasoning may be employed in the comparison of all other similar bodies, whether the bodies are bounded by planes or by curved surfaces, for the lines employed to measure the surfaces of all those bodies have always the same number of parts of their scales, and consequently the products of these lines contain the same number of times the squares of these same parts. And if the lines necessary for measuring the surface of similar bodies were incommensurable, it is clear that the demonstration would still apply on the principles employed (2nd Part, Par. 28), in comparing similar figures of which the sides are incommensurable.

81.

It may be proved, in the same way, that the surfaces of spheres are as the squares of their radii. But in order to see this more clearly in another way, it may suffice to remember that the surfaces of circles are as the squares of their radii

(3rd Part, Par. 6), and that the surfaces of spheres are four times those of their great circles (Par. 69).

82.

The proportionality between the surfaces of similar bodies and the squares of their homologous sides is so general, that it applies as much to bodies that cannot be measured as to those which can be measured.

Without knowing how to measure, for example, the surface of an oblique cylinder, we can affirm that the surfaces of two similar oblique cylinders are as the squares of the diameter of their bases; for on inscribing in these two cylinders two similar prisms, or as many as we please, we see, by what precedes, that the surfaces of these prisms are as the squares of the diameters of the bases. Therefore the cylinders themselves, considered as the ultimate inscribed prisms, have their surfaces in the same ratio.

83.

The fundamental proposition for the comparison of the

Fig. 177.

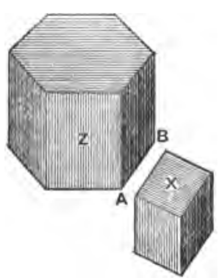
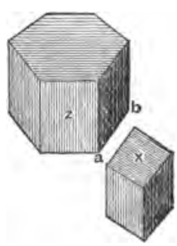


Fig. 178.



solidity of similar bodies is this:—Similar solids are to each other as the cubes of their homologous sides.

This proposition can be demonstrated, like the preceding, by considering that similar figures differ only in the scales on which they are constructed.

To make this apparent in the most simple manner, we may take, as an example, two similar prisms Z, and *z*, and two cubes X, and *x*, of which the sides are equal to A B and *a b*, corresponding lines in the two prisms, and we take also the two scales A B and *a b*, divided into a number of parts sufficiently great for measuring the dimensions of these solids. This being assumed, it is obvious that there are as many cubes made upon the parts of *a b*, in the prism *z*, and in the cube *x*, as there are cubes made on the parts of A B, in the prism Z, and in the cube X. The same reasoning is applicable to all other solids; and such as have incommensurable dimensions must also be in the same ratio as the cubes of their homologous sides.

84.

The solidities of spheres, for example, are evidently as the cubes of their radii.

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